What is The pressure distribution in a spherical cloud of ideal gas of mass M? (Assume self-gravitation)

use 3 equations:

1) Poisson's Eqn for gravity
2) Newton's force balance
3) Ideal gas law

FORCE BALANCE

$$\tau_{ij,j} + f_i = 0$$

$$(-p\delta_{ij})_{,j} + f_i = 0$$

$$-p_{,i} + f_i = 0$$

$$-\nabla p + f_i = 0$$

$$-\nabla p + \rho \underline{g} = 0$$

$$\boxed{-k\nabla\rho + \rho\underline{g} = 0}$$

IDEAL GAS LAW

$$pV = nRT$$

$$V = \frac{nRT}{p}$$

$$\rho = \frac{nm}{V} = \frac{nm}{nRT}p = \frac{m}{RT}p$$

$$p = k\rho \qquad k = \frac{RT}{m}$$

POISSON EQN

$$\nabla^2\Phi = 4\pi\gamma\rho = \nabla\cdot\nabla\Phi$$

$$\underline{g} = -\nabla\Phi$$

$$\boxed{\nabla\cdot\underline{g} = -4\pi\gamma\rho}$$

$$\nabla\rho = \frac{\partial\rho}{\partial r}$$

$$\nabla\cdot\underline{g} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 g_r)$$

$$\boxed{-k\frac{d\rho}{dr} + \rho g_r = 0 \qquad \frac{1}{r^2}\frac{d}{dr}(r^2 g_r) = -4\pi\gamma\rho}$$

$$g_r = +k\frac{1}{\rho}\frac{d\rho}{dr} = +k\frac{d}{dr}\ln\rho = +k\frac{d}{dr}q \qquad (q = \ln\rho)$$

$$+\frac{1}{r^2}\frac{d}{dr}\left(r^2 k\frac{dq}{dr}\right) = -4\pi\gamma e^{q} \quad \text{or} \quad \boxed{\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d\rho}{dr}\right) = -\frac{4\pi\gamma}{k}\rho}$$

FINAL EQN

Let $\begin{cases} c = 4\pi\gamma/k \\ y = p = \text{pressure} \\ x = r = \text{radius} \end{cases}$

Then: $\frac{1}{x^2}\frac{d}{dx} x^2 \frac{1}{y}\frac{dy}{dx} = -cy$

now write as 2 first order equations

if $\quad z \equiv x^2 \frac{1}{y}\frac{dy}{dx}$

Then $\quad \frac{dz}{dx} = -cx^2 y \quad \equiv f(x,y,z)$

and $\quad \frac{dy}{dx} = x^{-2} y z \quad \equiv g(x,y,z)$

now solve numerically using Runga Kutta method with initial conditions

pressure(0) = $y(x=0) = 1.0$

pressure gradient (0) = 0.0 implies $z(x=0) = 0$

25.5.14

$$\text{P:}\quad y_{n+1}=y_{n-5}+\frac{3h}{10}(11y'_n-14y'_{n-1}+26y'_{n-2}-14y'_{n-3}+11y'_{n-4})+O(h^7)$$

$$\text{C:}\quad y_{n+1}=y_{n-3}+\frac{2h}{45}(7y'_{n+1}+32y'_n+12y'_{n-1}+32y'_{n-2}+7y'_{n-3})+O(h^7)$$

Formulas Using Higher Derivatives

25.5.15

$$\text{P:}\quad y_{n+1}=y_{n-2}+3(y_n-y_{n-1})+h^2(y''_n-y''_{n-1})+O(h^5)$$

$$\text{C:}\quad y_{n+1}=y_n+\frac{h}{2}(y'_{n+1}+y'_n)-\frac{h^2}{12}(y''_{n+1}-y''_n)+O(h^5)$$

25.5.16

$$\text{P:}\quad y_{n+1}=y_{n-2}+3(y_n-y_{n-1})+\frac{h^3}{2}(y'''_n+y'''_{n-1})+O(h^7)$$

$$\text{C:}\quad y_{n+1}=y_n+\frac{h}{2}(y'_{n+1}+y'_n)-\frac{h^2}{10}(y''_{n+1}-y''_n)+\frac{h^3}{120}(y'''_{n+1}+y'''_n)+O(h^7)$$

Systems of Differential Equations

First Order: $y'=f(x,y,z),\ z'=g(x,y,z).$

Second Order Runge-Kutta

25.5.17

$$y_{n+1}=y_n+\frac{1}{2}(k_1+k_2)+O(h^3),\qquad z_{n+1}=z_n+\frac{1}{2}(l_1+l_2)+O(h^3)$$

$$k_1=hf(x_n,y_n,z_n),\qquad l_1=hg(x_n,y_n,z_n)$$

$$k_2=hf(x_n+h,y_n+k_1,z_n+l_1),\qquad l_2=hg(x_n+h,y_n+k_1,z_n+l_1)$$

Fourth Order Runge-Kutta

25.5.18

$$y_{n+1}=y_n+\frac{1}{6}(k_1+2k_2+2k_3+k_4)+O(h^5),\qquad z_{n+1}=z_n+\frac{1}{6}(l_1+2l_2+2l_3+l_4)+O(h^5)$$

$$k_1=hf(x_n,y_n,z_n)\qquad l_1=hg(x_n,y_n,z_n)$$

$$k_2=hf\left(x_n+\frac{1}{2}h,y_n+\frac{1}{2}k_1,z_n+\frac{1}{2}l_1\right)\qquad l_2=hg\left(x_n+\frac{h}{2},y_n+\frac{k_1}{2},z_n+\frac{l_1}{2}\right)$$

$$k_3=hf\left(x_n+\frac{1}{2}h,y_n+\frac{1}{2}k_2,z_n+\frac{1}{2}l_2\right)\qquad l_3=hg\left(x_n+\frac{h}{2},y_n+\frac{k_2}{2},z_n+\frac{l_2}{2}\right)$$

$$k_4=hf(x_n+h,y_n+k_3,z_n+l_3)\qquad l_4=hg(x_n+h,y_n+k_3,z_n+l_3)$$

Second Order: $y''=f(x,y,y')$

Milne's Method

25.5.19

$$\text{P:}\quad y'_{n+1}=y'_{n-3}+\frac{4h}{3}(2y''_{n-2}-y''_{n-1}+2y''_n)+O(h^5)$$

$$\text{C:}\quad y'_{n+1}=y'_{n-1}+\frac{h}{3}(y''_{n-1}+4y''_n+y''_{n+1})+O(h^5)$$

Runge-Kutta Method

25.5.20

$$y_{n+1}=y_n+h\left[y'_n+\frac{1}{6}(k_1+k_2+k_3)\right]+O(h^5)$$

$$y'_{n+1}=y'_n+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$k_1=hf(x_n,y_n,y'_n)$$

$$k_2=hf\left(x_n+\frac{1}{2}h,y_n+\frac{h}{2}y'_n+\frac{h}{8}k_1,y'_n+\frac{k_1}{2}\right)$$

$$k_3=hf\left(x_n+\frac{1}{2}h,y_n+\frac{h}{2}y'_n+\frac{h}{8}k_1,y'_n+\frac{k_2}{2}\right)\quad *$$

$$k_4=hf\left(x_n+h,y_n+hy'_n+\frac{h}{2}k_3,y'_n+k_3\right)$$

Second Order: $y''=f(x,y)$

Milne's Method

25.5.21

$$\text{P:}\quad y_{n+1}=y_n+y_{n-2}-y_{n-3}+\frac{h^2}{4}(5y''_n+2y''_{n-1}+5y''_{n-2})+O(h^6)$$

$$\text{C:}\quad y_n=2y_{n-1}-y_{n-2}+\frac{h^2}{12}(y''_n+10y''_{n-1}+y''_{n-2})+O(h^6)$$

Runge-Kutta Method

25.5.22 $$y_{n+1}=y_n+h\left(y'_n+\frac{1}{6}(k_1+2k_2)\right)+O(h^4)$$

$$y'_{n+1}=y'_n+\frac{1}{6}k_1+\frac{2}{3}k_2+\frac{1}{6}k_3$$

$$k_1=hf(x_n,y_n)$$

$$k_2=hf\left(x_n+\frac{h}{2},y_n+\frac{h}{2}y'_n+\frac{h}{8}k_1\right)$$

$$k_3=hf\left(x_n+h,y_n+hy'_n+\frac{h}{2}k_2\right).$$

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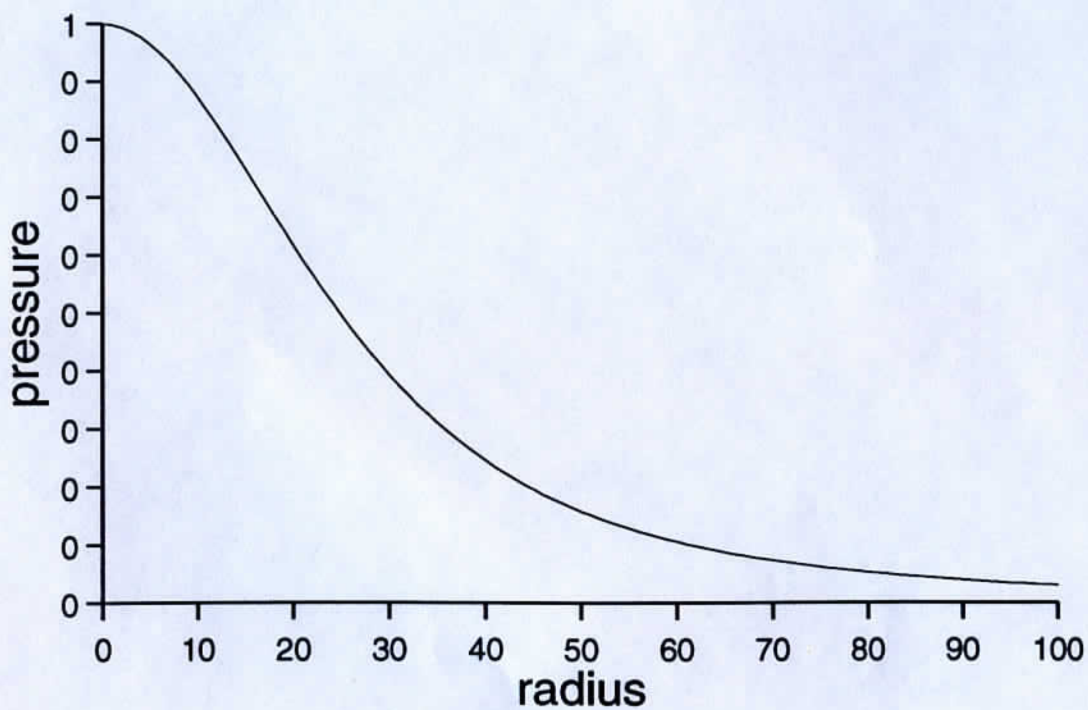
*See page II.

```c
#include <stdio.h>
#include <math.h>

double f(), g();
double c=0.01;

main()
{
        double x, y, z, h;
        double k1, k2, k3, k4;
        double l1, l2, l3, l4;

        int i;

        x = 0.0;
        y = 1.0;
        z = 0.0;
        h = 1.0;

        printf("%f %f\n", x, y);

        for( i=0; i< 100; i++ ) {

                k1 = h*f(x,y,z);
                l1 = h*g(x,y,z);
                k2 = h*f(x+h/2.0, y+k1/2.0, z+l1/2.0);
                l2 = h*g(x+h/2.0, y+k1/2.0, z+l1/2.0);
                k3 = h*f(x+h/2.0, y+k2/2.0, z+l2/2.0);
                l3 = h*g(x+h/2.0, y+k2/2.0, z+l2/2.0);
                k4 = h*f(x+h, y+k3, z+l3);
                l4 = h*g(x+h,y+k3,z+l3);

                y += (1.0/6.0)*(k1+2.0*k2+2.0*k3+k4);
                z += (1.0/6.0)*(l1+2.0*l2+2.0*l3+k4);
                x += h;
                printf("%f %f\n", x, y);
                }
        exit(0);
        }


double f( x, y, z )
double x, y, z;
{
        if( x<= 0.0 ) return (0.0 ); else return( ( y * z ) / ( x * x ) );
        }

double g( x, y, z )
double x, y, z;
{
        if( x <= 0.0 ) return( 0.0 ); else return ( - c * x * x * y );
        }
```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