MRN 141, April 2014

Frechet derivative of Traveltime with respect to waveform perturbation

after Marquering et al, GJI 137, 805-815, 1999

$u^{obs} = u + \delta u$

$d(\tau) = \int \underset{\text{reference}}{u(t-\tau)}\ \underset{\text{modeled}}{u_M(t)}\, dt$

$u_M(t) = u(t) + \delta u(t)$

modeled = reference + perturbation

$d(\tau) = c(\tau) + \delta c(\tau)$

$c(\tau) = \int u(t-\tau)\, u(t)\, dt$

$\delta c(\tau) = \int \underset{\text{reference}}{u(t-\tau)}\, \delta u(t)\, dt$

Diff Travel time δ ... cc.

max of cross corr.

$\delta u = 0 \Rightarrow$ max at $\tau = 0$

$\delta u \neq 0 \Rightarrow$ max at $\delta\tau \neq 0$

$$d(\delta\tau) = c(0) + \underbrace{\left(\frac{d}{d\tau} c\right)\Big|_{\tau=0}}_{\partial_\tau c(0)} \delta\tau + \underbrace{\frac{1}{2}\left(\frac{d^2}{d\tau^2} c\right)}_{\frac{1}{2}\partial_{\tau\tau} c(0)} \delta\tau^2 + \delta c(0) + \underbrace{\left(\frac{d}{d\tau}\delta c\Big|_{\tau=0}\right)}_{\partial_\tau \delta c(0)} \delta\tau$$

$$= c(0) + \underbrace{\partial_t c(0)\delta\tau}_{0 \text{ at max}} + \partial_{\tau\tau} c(0)\, \delta\tau^2 + \delta c(0) + \partial_\tau \delta c(0)\, \delta\tau$$

max when $\frac{\partial}{\partial \delta\tau} d(\delta\tau) = 0$

$$0 = \frac{\partial}{\partial \delta\tau}\left[c(0) + 0 + \frac{1}{2}\partial_{\tau\tau} c(0)\, \delta\tau^2 + \delta c(0) + \partial_\tau \delta c(0)\, \delta\tau \right]$$

$$= 0 + 0 + \partial_{\tau\tau} c(0)\, \delta\tau + 0 + \partial_\tau \delta c(0)$$

$$\delta\tau = -\frac{\partial_\tau \delta c(0)}{\partial_{\tau\tau} c(0)}$$

$$= \frac{d}{d\tau}\int u(t-\tau)\, \delta u(t)\, dt$$

$$\partial_\tau \delta c(\tau) = \int \dot u(t-\tau) \frac{d(t-\tau)}{d\tau}\, \delta u(t)\, dt$$

$$= -\int \dot u(t-\tau)\, \delta u(t)\, dt$$

$$\overset{\tau=0}{=} -\int \dot u(t)\, \delta u(t)\, dt$$

$$\partial_{\tau\tau} c(\tau) = \frac{d^2}{d\tau^2}\int u(t-\tau)\, u(t)\, dt$$

$$= \int \ddot u(t-\tau)\, u(t)\, dt$$

$$= A$$

$$\delta\tau = \left(\delta u, \frac{\dot u}{A}\right)$$