Energy Flux of Elastic Waves – Bill Menke – Oct 22, 2015

Fundamental relations:

$$\text{Newton's Law: } \rho \ddot{u}_i = \tau_{ij,j}$$

$$\text{Stress} - \text{Strain: } \tau_{ij} = c_{ijpq}\varepsilon_{pq} \text{ with } c_{ijpq} = c_{jipq} = c_{ijqp} = c_{pqij}$$

$$\text{Strain} - \text{Displacement: } \varepsilon_{pq} = ½\left(u_{p,q} + u_{q,p}\right)$$

Equation of Motion:

$$\rho \ddot{u}_i = c_{ijpq} u_{p,jq}$$

Energy density is sum of strain energy and kinetic energy:

$$E = ½\tau_{ij}\varepsilon_{ij} + ½\rho \dot{u}_i \dot{u}_i$$

$$= ½ c_{ijpq} u_{p,q} u_{i,,j} + ½\rho \dot{u}_i \dot{u}_i$$

Energy flux (rate of energy transport per unit area) in direction i (Synge, 1956-1957):

$$F_i = -\tau_{ij}\dot{u}_j$$

$$= -c_{ijpq} u_{p,q} \dot{u}_j$$

Conservation of energy

$$\dot{E} = -F_{i,i}$$

Proof that E and F_i obey conservation of energy (for constant c_{ijpq}):

$$\dot{E} = c_{ijpq} u_{p,q} \dot{u}_{i,,j} + \rho \dot{u}_i \ddot{u}_i$$

$$-F_{i,i} = \left(c_{ijpq} u_{p,q} \dot{u}_j\right)_{,i}$$

Apply chain rule:

$$-F_{i,i} = c_{ijpq} u_{p,q} \dot{u}_{j,i} + \left(c_{ijpq} u_{p,qi}\right)\dot{u}_j$$

Insert equation of motion:

$$-F_{i,i} = c_{ijpq} u_{p,q} \dot{u}_{j,i} + \rho \dot{u}_j \ddot{u}_j$$

Apply symmetry of c_{ijpq} and rename summation indices:

$$-F_{i,i} = c_{ijpq} u_{p,q} \dot{u}_{i,j} + \rho \dot{u}_i \ddot{u}_i = \dot{E}$$

For a real displacement, the positive and negative frequency components are complex conjugate pairs:

$$u_i = U_i \ \exp(-i\omega t) + \bar{U}_i \ \exp(+i\omega t)$$

$$= 2U_i^R \ \cos(\omega t) + \ 2U_i^I \ \sin(\omega t)$$

Here the overbar indicates complex conjugation and U_i^R and U_i^I are the real and imaginary parts of U_i, respectively. The corresponding real stress is:

$$\tau_{ij} = T_{ij} \ \exp(-i\omega t) + \bar{T}_{ij} \ \exp(+i\omega t)$$

$$= 2T_{ij}^R \cos(\omega t) + 2T_{ij}^I \ \sin(\omega t)$$

Note that these quantities satisfy the stress-strain relation $T_{ij} = c_{ijpq} U_{p,q}$ and the equation of motion $-\rho\omega^2 U_i = c_{ijpq} U_{j,pq}$. Inserting into the flux equation yields:

$$F_i = -\tau_{ij} \dot{u}_j$$

$$= -4\omega\{T_{ij}^R \cos(\omega t) + T_{ij}^I \ \sin(\omega t)\} \ \{-U_j^R \ \sin(\omega t) + \ U_j^I \ \ \cos(\omega t)\}$$

$$= -4\omega\{T_{ij}^R \ U_j^I \cos^2(\omega t) - T_{ij}^I \ U_j^R \ \sin^2(\omega t) + T_{ij}^I (U_j^I - T_{ij}^R U_j^R) \cos(\omega t) \sin(\omega t)\}$$

We time-average the trigonometric functions $\langle\cos^2(\omega t)\rangle = \ \langle\sin^2(\omega t)\rangle = \ $ ½ and $\langle\cos(\omega t) \sin(\omega t)\rangle = \ 0$, where $\langle . \rangle$ signified the average over one cycle. The time averaged flux is then:

$$\langle F_i \rangle = -2\omega\{T_{ij}^R U_j^I - T_{ij}^I \ U_{ij}^R\}$$

$$=-2\omega c_{ijpq}\{U_{p,q}^R U_j^I - U_{p,q}^I \ U_j^R\}$$

In the isotropic case, $c_{ijpq} = \lambda\delta_{ij}\delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{jp}\delta_{iq})$, so the average flux is:

$$\frac{\langle F_i \rangle}{-2\omega} = \lambda(U_{p,p}^R U_i^I - U_{p,p}^I U_i^R) + \mu(U_{i,q}^R U_q^I - U_{i,q}^I U_q^R) + \mu(U_{j,i}^R U_j^I - U_{j,i}^I U_j^R)$$

For $i = 1$ and $U_2^R = 0$ and $U_{p,2}^R = 0$ (for all p):

$$\frac{\langle F_1 \rangle}{-2\omega} = (\lambda + 2\mu)(U_{1,1}^R U_1^I - U_{1,1}^I U_1^R) +$$

$$+\mu\left(U^R_{3,3}U^I_1 + U^R_{1,3}U^I_3 + U^R_{3,1}U^I_3 - U^I_{3,3}U^R_1 - U^I_{1,3}U^R_3 - U^I_{3,1}U^R_3\right)$$

Now suppose:

$$U_i = p_i(z)\ \exp(\pm ikx)$$

$$U_1 = p_1\ \exp(\pm ikx) \quad \text{and} \quad U_3 = p_3\ \exp(\pm ikx)$$

$$U_{1.1} = \pm ik\, p_1 \exp(\pm ikx) \quad \text{and} \quad U_{3,1} = \pm ik\, p_3 \exp(\pm ikx)$$

$$U_{1.3} = p_{1.3}\ \exp(\pm ikx) \quad \text{and} \quad U_{3,3} = p_{3,3}\ \exp(\pm ikx)$$

The vertically integrated horizontal flux $\langle F_i \rangle_T$ can be calculated as:

$$\langle F_1 \rangle_T = \int_0^\infty \langle F_1 \rangle\ dz$$

Reference

Synge, J.L., Flux of energy for elastic waves in anisotropic media, Proceedings of the Royal Irish Academy: Section A, Mathematical and Physcial Sciences, Volume 58, 1956-1957, 12-21.