Force equivalent to an initial condition for a second order linear differential equation

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1. We consider the 2^{nd} order linear differential equation:

$$\mathcal{L}y = \left\{\frac{d}{dt}p(t)\frac{d}{dt} + p(t)\right\}yp(t) = f(t)$$

defined on the interval (T_1, T_2). The Green function $G(t,\tau)$,satisfies $\mathcal{L}G(t,\tau) = \delta(t-\tau)$ and homogenous boundary conditions.

2. $\mathcal{L}$ is self-adjoint, as

$$\left(\frac{d}{dt}p(t)\frac{d}{dt} + p(t)\right)^{\dagger} = \left(\frac{d}{dt}\right)^{\dagger}\left(p(t)\right)^{\dagger}\left(\frac{d}{dt}\right)^{\dagger} + \left(p(t)\right)^{\dagger} = \frac{d}{dt}p(t)\frac{d}{dt} + p(t)$$

as d/dt is anti-self-adjoint and $p(t)$ is self-adjoint. It follows that

$$(v, \mathcal{L}u)_t = (u, \mathcal{L}v)_t \quad \text{with} \quad (a,b)_t = \int_{T_1}^{T_2} ab\, dt$$

for any two solutions, u and v.

3. The Green function satisfies the reciprocity relationship $G(t,\tau) = G(\tau,t)$. Inserting the Green function for source at τ_1 and τ_2:

$$\left(G(t,\tau_2), \delta(t-\tau_1)\right)_t = \left(G(t,\tau_1), \delta(t-\tau_2)\right)_t$$

$$G(\tau_1,\tau_2) = G(\tau_2,\tau_1)$$

4. Green's identity for two solutions, u and v is

$$\int_{T_1}^{T_2} (u\,\mathcal{L}v - v\,\mathcal{L}u)\, dt = (puv_t - pu_t v)_{T_1}^{T_2}$$

with $u_t \equiv du/dt$ and $v_t \equiv dv/dt$. It is derived by applying integration by parts

$$\int_{T_1}^{T_2} (u\,\mathcal{L}v - v\,\mathcal{L}u)\, dt =$$

$$\int_{T_1}^{T_2} \left(u\,\frac{d}{dt}pv_t + upv - v\,\frac{d}{dt}pu_t - vpu\right) dt$$

$$= \int_{T_1}^{T_2} \left(u\,\frac{d}{dt}(pv_t) - v\,\frac{d}{dt}(pu_t)\right) dt$$

$$= (puv_t - pu_tv)|_{T_1}^{T_2} - \int_{T_1}^{T_2} (pu_tv_t - pu_tv_t)\, dt$$

$$= (puv_t - pu_tv)|_{T_1}^{T_2}$$

5. A formula for the solution involving initial conditions:

$$y(t = T_1) \quad \text{and} \quad y_t(t = T_1)$$

$$y(t = T_2) = 0 \quad \text{and} \quad y_t(t = T_2) = 0$$

is obtained by equating $u = y$ and $v = G(t, \tau)$

$$\int_{t=T_1}^{t=T_2} (u\, \mathcal{L}v - v\, \mathcal{L}u)\, dt = p(t)\big(u(t)v_t(t) - u_t(t)v(t)\big)\Big|_{t=T_1}^{t=T_2}$$

$$\int_{t=T_1}^{t=T_2} \big(y\, \delta(t-\tau) - G(t,\tau)\, f(t)\big)\, dt = p(t)[y(t)G_t(t,\tau) - y_tG(t,\tau)]|_{t=T_1}^{t=T_2}$$

$$y(\tau) - \int_{t=T_1}^{t=T_2} G(t,\tau)\, f(t)\, dt = p(t)[y(t)G_t(t,\tau) - y_tG(t,\tau)]|_{t=T_1}^{t=T_2}$$

$$y(\tau) = \int_{t=T_1}^{t=T_2} G(t,\tau)\, f(t)\, dt + p(t)[y(t)G_t(t,\tau) - y_tG(t,\tau)]|_{t=T_1}^{t=T_2}$$

$$y(\tau) = \int_{t=T_1}^{t=T_2} G(t,\tau)\, f(t)\, dt - p(t)[y_tG(t,\tau) - y(t)G_t(t,\tau)]|_{t=T_1}^{t=T_2}$$

Now swap the names of t and τ

$$y(t) = \int_{\tau=T_1}^{\tau=T_2} G(\tau,t)\, f(\tau)\, d\tau - p(\tau)[y_\tau G(\tau,t) - y(\tau)G_\tau(\tau,t)]|_{\tau=T_1}^{\tau=T_2}$$

Apply reciprocity, noting $G_\tau(\tau,t) = \frac{\partial}{\partial\tau}G(\tau,t) = \frac{\partial}{\partial\tau}G(t,\tau)$

$$y(t) = \int_{\tau=T_1}^{\tau=T_2} G(t,\tau)\, f(\tau)\, d\tau - p(\tau)[y_\tau G(t,\tau) - y(\tau)G_\tau(\tau,t)]|_{\tau=T_1}^{\tau=T_2}$$

$$y(t) = \int_{\tau=T_1}^{\tau=T_2} G(t,\tau)\, f(\tau)\, d\tau - p(\tau)\left[y_\tau(\tau)G(t,\tau) - y(\tau)\frac{\partial G(t,\tau)}{\partial\tau}\right]\Bigg|_{\tau=T_1}^{\tau=T_2}$$

This results agrees with (12.1.5) of Copley (2015). By assumption, only the lower boundary condition contributes

$$y(t) = \int_{\tau=T_1}^{\tau=T_2} G(t,\tau)\, f(\tau)\, d\tau + p(\tau = T_1)\left[y_\tau(\tau = T_1)G(t,\tau = T_1) - y(\tau = T_1)\frac{\partial G(t,\tau = T_1)}{\partial\tau}\right]$$

6. Inserting the force

$$f(\tau) = A\delta\big(\tau - (T_1 + \varepsilon)\big) + B\frac{\partial}{\partial\tau}\delta\big(\tau - (T_1 + \varepsilon)\big)$$

with $\varepsilon \ll (T_2 - T_1)$ into the Green function integral, and integrating

$$\int_{\tau=T_1}^{\tau=T_2} G(t,\tau)\, f(\tau)\, d\tau = AG(t,T_1) - B\frac{\partial G(t,\tau = T_1)}{\partial\tau}$$

7. With the choices

$$A = p(\tau = T_1)y_\tau(\tau = T_1) \quad \text{and} \quad B = p(\tau = T_1)y(\tau = T_1)$$

the force is equivalent to the boundary condition. Thus the boundary conditions at T_1 can be replaced by a virtual force acting at time $T_1 + \varepsilon$

Copley, L., 2015, Chapter 12, Non-Homogeneous Boundary Value Problems: Green's Functions, in Mathematics for the Physical Sciences, De Gruyter Open Poland, www.degruyter.com/document/doi/10.2478/9783110409475.12/html