Sensitivity Kernel for Finite Frequency Double Differential Travel Times

Bill Menke, March 21, 2017

Loosely based on Yuan, Simons & Tromp, Geophys. J. Int. 206, 1599-1618, 2016.

Method 1: The double difference is:

the differential between observed and modeled phase A
minus
the differential between observed and modeled phase B

At station A, the differential travel time $\Delta t_A^{obs,0} = t_A^{obs} - t_A^0$ between a observed windowed filtered seismogram $\Omega_A(t)\left[f(t) * u_A^{obs}(t)\right]$ and modeled windowed filtered seismogram $\Omega_A(t)[f(t) * u_A^0(t)]$ is measured using the cross-correlation method. Here $\Omega_A(t)$ is a window function, $f(t)$ is a filter and * signifies convolution. At station B, a corresponding differential travel time δt_B is measured. The two differential travel times are difference to yield:

$$\Delta\Delta t_{AB}^{obs,0} = \Delta t_A^{obs,0} - \Delta t_B^{obs,0} = t_A^{obs} - t_A^0 - t_B^{obs} + t_B^0$$

The modeled seismogram at station A is perturbed to $u_A = u_A^0 + \delta u_A$, and similarly at station B. The differential travel time changes from $\Delta t_A^{obs,0}$ to $\Delta t_A^{obs,0} + \delta\Delta t_A^{obs,0}$ (Marquering et al. 1999):

$$\delta\Delta t_A^{obs,0} = \int h_A^{obs}(t)\ \Omega_A(t)\ [f(t) * \delta u_A(t)]\, dt \quad \text{with}$$

$$h_A^{obs}(t) \equiv \frac{\dot{q}_A^{obs}\left(t - \Delta t_A^{obs,0}\right)}{N_A^{obs}} \quad \text{and} \quad q_A^{obs}(t) \equiv \Omega_A(t)\left[f(t) * u_A^{obs}(t)\right]$$

$$\text{and} \quad N_A^{obs} \equiv \int q_A^{obs}\ \ddot{q}_A^{obs}\, dt$$

and similarly for $\delta\Delta t_B^{obs,0}$. The time shift in the definition of $h_A^{obs}(t)$ aligns u_A^{obs} and $u_A^0(t)$; the time shift associated with the perturbation δu_A is measured with respect to this best alignment. These time integrals can be promoted to time/space inner products $(.,.)$ by introducing spatial Dirac impulse functions:

$$\delta\Delta t_A^{obs,0} = \left(h_A^{obs}\ \delta(x - x_A), \Omega_A(f * \delta u)\right) = \left(f \star \left(\Omega_A h_A^{obs}\right) \delta(x - x_A), \delta u\right)$$

$$= \left(H_A^{obs}(t)\delta(x - x_A), \delta u\right) \quad \text{with} \quad H_A^{obs}(t) \equiv f \star \left(\Omega_A h_A^{obs}\right)$$

Here $\star$ signifies cross-correlation. The difference between the two perturbations is:

$$\delta\Delta\Delta t_{AB}^{obs,0} \equiv \left(H_A^{obs}(t)\delta(x-x_A) - H_B^{obs}(t)\delta(x-x_B), \delta u\right) = (s^{obs}(x,t), \delta u)$$

$$\text{with} \quad s^{obs}(x,t) \equiv H_A^{obs}(t)\delta(x-x_A) - H_B^{obs}(t)\delta(x-x_B)$$

The Born approximation indicates that a change δm in material properties m results in a change δu in the wave field:

$$\delta u = -\mathcal{L}^{-1}\frac{\partial \mathcal{L}}{\partial m} u_0 \, \delta m$$

Substituting the equation into the inner product yields:

$$\delta\Delta\Delta t_{AB}^{obs,0} = -\left(s(x,t), \mathcal{L}^{-1}\frac{\partial \mathcal{L}}{\partial m} u_0\right)\delta m = -\left(\mathcal{L}^{\dagger -1} s(x,t), \frac{\partial \mathcal{L}}{\partial m} u_0\right)\delta m$$

$$= -\left(\lambda, \frac{\partial \mathcal{L}}{\partial m} u_0\right)\delta m \quad \text{with} \quad \mathcal{L}^{\dagger}\lambda = s^{obs}(x,t)$$

Here $\lambda(x,t)$ is an adjoint field that satisfies the adjoint wave equation. Finally, consider m to be associated with point density heterogeneity $\rho = \rho_0 + m\delta(x - x_H)$. The part of the wave field operator involving density is $\mathcal{L} = \rho\, \partial^2/\partial t^2$. Then:

$$\frac{\partial \mathcal{L}}{\partial m} = \delta(x - x_H)\frac{\partial^2}{\partial t^2}$$

We can perform the spatial integral in the inner product, yielding a formula for the data kernel:

$$\frac{\partial}{\partial m}\Delta\Delta t_{AB}^{obs,0} = \frac{\delta\Delta\Delta t_{AB}^{obs,0}}{\delta m} = -\int \lambda(x_H,t)\, \ddot{u}_0(x_H,t)\, dt$$

$$\mathcal{L}^{\dagger}\lambda = s^{obs}(x,t)$$

The derivative can be calculated via a single adjoint field that has two sources, located at each of the two stations. In the special case of differential times between two phases observed on the same station (for example $SS - S$ differential times), $x_A = x_B$ but $\Omega_A \neq \Omega_B$; only a single source is needed.

Method 2: The double difference is:

the differential between observed phases A and B
minus
the differential between synthetic phases A and B

The two differential travel times are difference to yield:

$$\Delta\Delta t_{AB}^{obs,0} = \Delta t_{AB}^{obs} - \Delta t_{AB}^{0} = t_A^{obs} - t_B^{obs} - t_A^{0} + t_B^{0}$$

A perturbation in the model leads only to a change in Δt_{AB}^{0}, and not Δt_{AB}^{obs}:

$$\delta\Delta\Delta t_{AB}^{obs,0} = -\delta\Delta t_{AB}^{0}$$

However, a perturbation δm in the model changes leads to both a perturbation in δu_A and δu_B, and each of these contribute to $\delta\Delta t_{AB}^{0}$:

$$\delta\Delta\Delta t_{AB}^{obs,0} = -\delta\Delta t_{AB}^{0}$$

$$= \int h_A^0(t)\ \Omega_B(t)\left(f(t) * \delta u_B(t)\right) dt - \int h_B^0(t)\ \Omega_A(t)\left(f(t) * \delta u_A(t)\right) dt$$

$$h_A^0(t) = \frac{\dot{q}_A(t-\Delta t_{AB}^0)}{N_A} \text{ and } q_A(t) = \Omega_A(t)\,[f(t) * u_A^0(t)]$$

$$\text{and } N_A = \int q_A\ \ddot{q}_A\ dt$$

As before, the time integrals are promoted to inner products though the introduction of Dirac impulse functions and these inner products are manipulated using adjoint methods:

$$\delta\Delta\Delta t_{AB}^{obs,0} = (h_A^0\delta(x-x_B), \Omega_B\ (f * \delta u)\ ) - (h_B^0(t)\delta(x-x_A), \Omega_A\ (f * \delta u)\ )$$

$$= (f \star (\Omega_B h_A^0)\delta(x-x_A), \delta u\ ) - (f \star (\Omega_A h_B^0)\ \delta(x-x_A), \delta u\ )$$

Or:

$$\Delta\delta t_{AB} = (s^0(x,t), \delta u) \quad \text{with} \quad s^0(x,t) = f \star (\Omega_A h_A^0)\ \delta(x-x_A) - f \star (\Omega_B h_B^0)\delta(x-x_A)$$

After substituting the Born approximation and the formula for a point density heterogeneity, we find:

$$\frac{\partial}{\partial m}\Delta\Delta t_{AB}^{obs,0} = \frac{\delta\Delta\Delta t_{AB}^{obs,0}}{\delta m} = -\int \lambda(x_H,t)\ \ddot{u}_0(x_H,t)\ dt$$

$$\mathcal{L}^{\dagger}\lambda = s^0(x,t)$$

For any set of observed data $\mathbf{d}^{\mathrm{obs}}$ and predicted data $\mathbf{d}$, the total error E is defined as:

$$E = \mathbf{e}^T\mathbf{e} \text{ with } \mathbf{e} = \mathbf{d}^{\mathrm{obs}} - \mathbf{d}$$

The error derivative is:

$$\frac{\partial E}{\partial m} = 2\mathbf{e}^T\frac{\partial \mathbf{e}}{\partial \mathrm{m}} = -2\mathbf{e}^T\frac{\partial \mathbf{d}}{\partial \mathrm{m}}$$

In the present case, the elements of $\mathbf{d}$ are the double differential travel times.

Example.

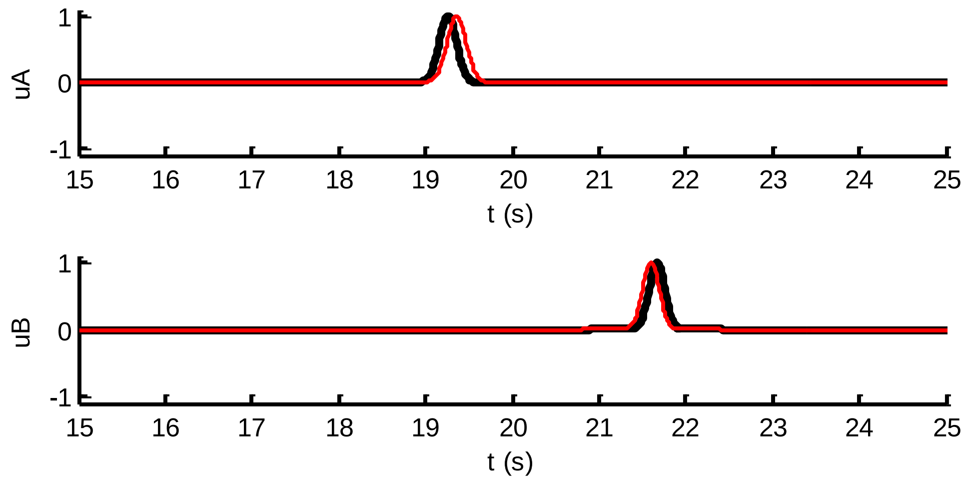

Figure 1: Observed pulses u_A^{obs} and u_B^{obs} (black) and unperturbed modeled pulses u_A^0 and u_B^0 (red).

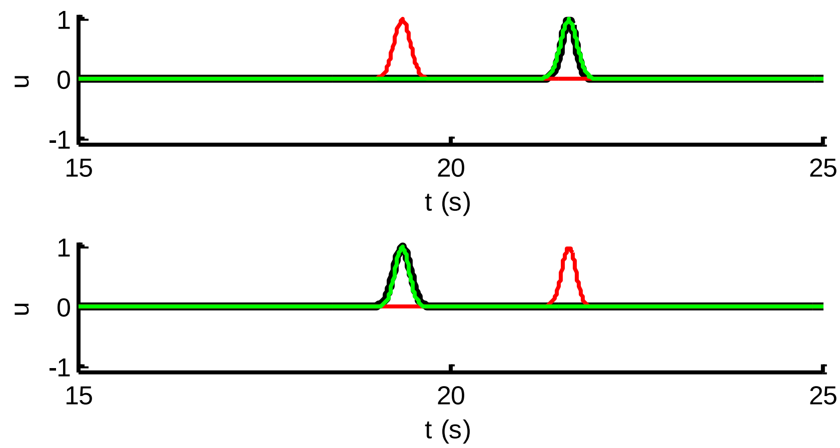

Figure 2: Aligned pulses. (Top) Modeled pulses $u_A^0(t)$ (red) $u_B^0(t)$ (black) and $u_A^0(t+\Delta t_{AB}^0)$ (green) bottom) Modeled pulses $u_A^0(t)$ (black) $u_B^0(t)$ (red) and $u_A^0(t+\Delta t_{BA}^0)$ (green).

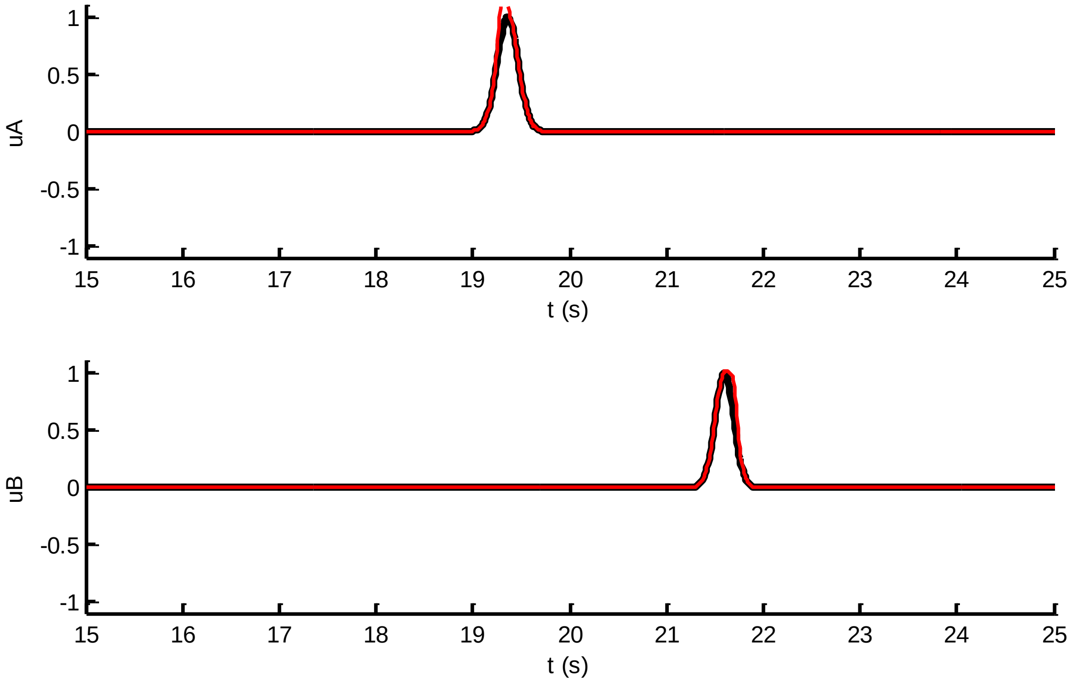

Figure 2 Unperturbed model pulses $u_A^0(t)$ and $u_A^0(t)$ (black) and perturbed model pulses $u_B(t)$ and $u_B(t)$ (red).

Results of test using doubledifference” code. Method 1 is referred to by “Different types” and Method 2 as “Same types”. Time differences are computed by cross-correlation (“xcorr”).

Unperturbed Case
Double difference time
Exact -0.170000
xcorr same types -0.170000
xcorr different types -0.170000

Perturbed Case
Double difference time
xcorr same types -0.140000
xcorr differents types -0.140000

Perturbation in differential time
xcorr same types   -0.030000
xcorr different types -0.030000
Marquering et al same -0.029435
Marquering et al diff -0.028336

```
clear all;
% Exercizes double differnce formulas
% Currently not attached to wave propagation model
% Instead, the perturbation in waveform du(t)
% is imposed ad hoc. Bill Menke, March 22, 2017

% time setup
Dt = 0.01;
N=4096;
t = Dt*[0:N-1]';

% wave forms are all Gaussian pulses
NAobs = floor(8*N/17);
tAobs = t(NAobs);
sigmaAobs = 10*Dt;
tA = tAobs+10*Dt;
sigmaA = 11*Dt;
NBobs = floor(9*N/17);
tBobs = t(NBobs);
sigmaBobs = 9*Dt;
tB = tBobs - 7*Dt;
sigmaB = 9.5*Dt;
uAobs = exp(-((t-tAobs).^2)/(2*(sigmaAobs^2)));
uA = exp(-((t-tA).^2)/(2*(sigmaA^2)));
uBobs = exp(-((t-tBobs).^2)/(2*(sigmaBobs^2)));
uB = exp(-((t-tB).^2)/(2*(sigmaB^2)));

fprintf('Unperturbed Case\n');
fprintf('Double difference time\n');
fprintf('Exact %f\n', (tAobs-tBobs)-(tA-tB) );

figure(1)
clf;
subplot(2,1,1);
set(gca,'LineWidth',2);
hold on;
axis( [15, 25, -1.1, 1.1] );
plot( t, uAobs, 'k-', 'LineWidth', 3 );
plot( t, uA, 'r-', 'LineWidth', 2 );
xlabel('t (s)');
ylabel('uA');
subplot(2,1,2);
set(gca,'LineWidth',2);
hold on;
axis( [15, 25, -1.1, 1.1] );
plot( t, uBobs, 'k-', 'LineWidth', 3 );
plot( t, uB, 'r-', 'LineWidth', 2 );
xlabel('t (s)');
```

```
ylabel('uB');

% cross correlate "same types" to find time lag
c = xcorr(uBobs, uAobs);
Nc = length(c);
Ncenter = (Nc+1)/2;
[cmax, icmax] = max(c);
tABobs = -Dt * (icmax-N);
c = xcorr(uB, uA);
Nc = length(c);
Ncenter = (Nc+1)/2;
[cmax, icmax] = max(c);
tAB = -Dt * (icmax-N);
fprintf('xcorr same types %f\n', tABobs-tAB );

% align "same"
uA2B = circshift( uA, (icmax-N) );
uB2A = circshift( uB, -(icmax-N) );
figure(2)
clf;
subplot(2,1,1);
set(gca,'LineWidth',2);
hold on;
axis( [15, 25, -1.1, 1.1] );
plot( t, uB, 'k-', 'LineWidth', 3 );
plot( t, uA, 'r-', 'LineWidth', 2 );
plot( t, uA2B, 'g-', 'LineWidth', 2 );
xlabel('t (s)');
ylabel('u');
subplot(2,1,2);
set(gca,'LineWidth',2);
hold on;
axis( [15, 25, -1.1, 1.1] );
plot( t, uA, 'k-', 'LineWidth', 3 );
plot( t, uB, 'r-', 'LineWidth', 2 );
plot( t, uB2A, 'g-', 'LineWidth', 2 );
xlabel('t (s)');
ylabel('u');

% cross correlate 'different types' to find time lag
c = xcorr(uA, uAobs);
Nc = length(c);
Ncenter = (Nc+1)/2;
[cmax, icmax] = max(c);
tAA = -Dt * (icmax-N);
iAA = (icmax-N);
c = xcorr(uB, uBobs);
Nc = length(c);
Ncenter = (Nc+1)/2;
```

```
[cmax, icmax] = max(c);
tBB = -Dt * (icmax-N);
iBB = (icmax-N);
fprintf('xcorr different types %f\n', tAA-tBB );
fprintf('\n');

% align "different"
uAobs2A = circshift( uAobs, iAA );
uBobs2B = circshift( uBobs, iBB );
figure(4)
clf;
subplot(2,1,1);
set(gca,'LineWidth',2);
hold on;
axis( [15, 25, -1.1, 1.1] );
plot( t, uA, 'k-', 'LineWidth', 3 );
plot( t, uAobs, 'r-', 'LineWidth', 2 );
plot( t, uAobs2A, 'g-', 'LineWidth', 2 );
xlabel('t (s)');
ylabel('u');
subplot(2,1,2);
set(gca,'LineWidth',2);
hold on;
axis( [15, 25, -1.1, 1.1] );
plot( t, uB, 'k-', 'LineWidth', 3 );
plot( t, uBobs, 'r-', 'LineWidth', 2 );
plot( t, uBobs2B, 'g-', 'LineWidth', 2 );
xlabel('t (s)');
ylabel('u');

fprintf('Perturbed Case\n');
fprintf('Double difference time\n');
dtA = -5*Dt;  % tuneable
dtB = 10*Dt;
dsigmaA = 4*Dt;
dsigmaB = 4*Dt;
duA = 0.3*exp(-((t-(tA+dtA)).^2)/(2*(dsigmaA^2)));
duB = 0.3*exp(-((t-(tB+dtB)).^2)/(2*(dsigmaB^2)));

% cross correlate "same types" to find time lag
c = xcorr(uBobs, uAobs);
Nc = length(c);
Ncenter = (Nc+1)/2;
[cmax, icmax] = max(c);
tABobsn = -Dt * (icmax-N);
c = xcorr(uB+duB, uA+duA);
Nc = length(c);
Ncenter = (Nc+1)/2;
[cmax, icmax] = max(c);
```

```
tABn = -Dt * (icmax-N);
fprintf('xcorr same types %f\n', tABobsn-tABn );

% cross correlate 'different types' to find time lag
c = xcorr(uA+duA, uAobs);
Nc = length(c);
Ncenter = (Nc+1)/2;
[cmax, icmax] = max(c);
tAAn = -Dt * (icmax-N);
c = xcorr(uB+duB, uBobs);
Nc = length(c);
Ncenter = (Nc+1)/2;
[cmax, icmax] = max(c);
tBBn = -Dt * (icmax-N);
fprintf('xcorr different types %f\n', tAAn-tBBn );
fprintf('\n');

figure(3)
clf;
subplot(2,1,1);
set(gca,'LineWidth',2);
hold on;
axis( [15, 25, -1.1, 1.1] );
plot( t, uA, 'k-', 'LineWidth', 3 );
plot( t, uA+duA, 'r-', 'LineWidth', 2 );
xlabel('t (s)');
ylabel('uA');
subplot(2,1,2);
set(gca,'LineWidth',2);
hold on;
axis( [15, 25, -1.1, 1.1] );
plot( t, uB, 'k-', 'LineWidth', 3 );
plot( t, uB+duB, 'r-', 'LineWidth', 2 );
xlabel('t (s)');
ylabel('uB');

% formula from Marquering et al, GJI 137, 805-815, 1999
% applied to "same" case
ref = uB2A;
per = duA;
refd = [diff(ref); 0]/Dt;  % 1st derivative
refdd = [0; diff(ref,2); 0]/(Dt^2);  % 2nd derivative
tau1 = (Dt*refd'*per)/(Dt*refdd'*ref); % estimate in delay

ref = uA2B;
per = duB;
refd = [diff(ref); 0]/Dt;  % 1st derivative
refdd = [0; diff(ref,2); 0]/(Dt^2);  % 2nd derivative
tau2 = (Dt*refd'*per)/(Dt*refdd'*ref); % estimate in delay
```

```
% applied to "different" case
ref = uAobs2A;
per = duA;
refd = [diff(ref); 0]/Dt;  % 1st derivative
refdd = [0; diff(ref,2); 0]/(Dt^2);  % 2nd derivative
tau3 = (Dt*refd'*per)/(Dt*refdd'*ref); % estimate in delay

ref = uBobs2B;
per = duB;
refd = [diff(ref); 0]/Dt;  % 1st derivative
refdd = [0; diff(ref,2); 0]/(Dt^2);  % 2nd derivative
tau4 = (Dt*refd'*per)/(Dt*refdd'*ref); % estimate in delay

fprintf('Perturbation in differential time\n');
fprintf('xcorr same types      %f\n', (tABn-tABobsn)-(tAB-tABobs) );
fprintf('xcorr different types %f\n', (tBBn-tAAn)-(tBB-tAA) );
fprintf('Marquering et al same %f\n', tau1-tau2 );
fprintf('Marquering et al diff %f\n', tau3-tau4 );
fprintf('\n');
```