A Data Resolution Matrix for Generalized Least Squares

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This derivation parallels the one for the model resolution matrix that's in Section 5.3 of Menke (2018).

The standard General Least Squares setup is:

$$\mathbf{Fm} = \mathbf{f} \;\text{ with }\; \mathbf{F} = \begin{bmatrix} \mathbf{C}_d^{-½}\mathbf{G} \\ \mathbf{C}_h^{-½}\mathbf{H} \end{bmatrix} \quad \text{and} \quad \mathbf{f}^{\text{obs}} = \begin{bmatrix} \mathbf{C}_d^{-½}\mathbf{d}^{\text{obs}} \\ \mathbf{C}_h^{-½}\mathbf{h}^{\text{pri}} \end{bmatrix} \tag{1}$$

The solution is $\mathbf{m}^{\text{est}} = [\mathbf{F}^{\text{T}}\mathbf{F}]^{-1}\mathbf{F}^{\text{T}}\mathbf{f}^{\text{obs}}$. According to Menke (2018, their Eqn. 5.48), this solution is equivalent to;

$$\mathbf{m}^{\text{est}} = \mathbf{F}^{-g}\mathbf{f}^{\text{obs}} = [\mathbf{A}^{-1}\mathbf{G}^{\text{T}}\mathbf{C}_d^{-1} \quad \mathbf{A}^{-1}\mathbf{H}^{\text{T}}\mathbf{C}_h^{-1}] \begin{bmatrix} \mathbf{d}^{\text{obs}} \\ \mathbf{h}^{\text{pri}} \end{bmatrix} \tag{2}$$

with $\mathbf{A} \equiv \mathbf{F}^{\text{T}}\mathbf{F} = \mathbf{G}^{\text{T}}\mathbf{C}_d^{-1}\mathbf{G} + \mathbf{H}^{\text{T}}\mathbf{C}_d^{-1}\mathbf{H}$.

Let the model parameters $\mathbf{m}^{(\text{H})}$ be those predicted on the basis of the prior information alone:

$$\mathbf{m}^{(\text{H})} = [\mathbf{H}^{\text{T}}\mathbf{C}_h^{-1}\mathbf{H}]^{-1}\mathbf{H}^{\text{T}}\mathbf{C}_h^{-1}\mathbf{h}^{\text{pri}} \tag{3}$$

Following Menke (2018), we assume that the matrix inverse exists; it can always be made to exist by adding very weak smallness information, if necessary. These model parameters predict data $\mathbf{d}^{(\text{H})} = \mathbf{Gm}^{(\text{H})}$, which we will call the "prior data" Now define two deviations, the deviation of the observed data from the prior data $\Delta\mathbf{d}^{\text{obs}} \equiv \mathbf{d}^{\text{obs}} - \mathbf{d}^{(\text{H})}$ and the deviations of the predicted data from the prior data $\Delta\mathbf{d}^{\text{pre}} \equiv \mathbf{d}^{\text{pre}} - \mathbf{d}^{(\text{H})}$. We will show that a data resolution matrix $\mathbf{N}$ can be defined such that $\Delta\mathbf{d}^{\text{pre}} = \mathbf{N}\,\Delta\mathbf{d}^{\text{obs}}$, and furthermore, that:

$$\mathbf{N} = \mathbf{GA}^{-1}\mathbf{G}^{\text{T}}\mathbf{C}_d^{-1} \tag{4}$$

Proof. Start with the predicted data $\mathbf{d}^{\text{pre}} = \mathbf{Gm}^{\text{est}}$ and insert the form of $\mathbf{m}^{\text{est}}$ given in (2).

$$\mathbf{d}^{\text{pre}} = \mathbf{GA}^{-1}\mathbf{G}^{\text{T}}\mathbf{C}_d^{-1}\mathbf{d}^{\text{obs}} + \mathbf{GA}^{-1}\mathbf{H}^{\text{T}}\mathbf{C}_h^{-1}\mathbf{h}^{\text{pri}}$$

Subtract $\mathbf{d}^{(\text{H})}$ from both sides, to obtain a deviation:

$$\Delta\mathbf{d}^{\text{pre}} \equiv \mathbf{d}^{\text{pre}} - \mathbf{d}^{(H)} = \mathbf{GA}^{-1}\mathbf{G}^{\text{T}}\mathbf{C}_d^{-1}\mathbf{d}^{\text{obs}} + \mathbf{GA}^{-1}\mathbf{H}^{\text{T}}\mathbf{C}_h^{-1}\mathbf{h}^{\text{pri}} - \mathbf{d}^{(\text{H})}$$

Replace $\mathbf{d}^{(\text{H})}$ with $\mathbf{Gm}^{(\text{H})}$ and insert (3):

$$\Delta\mathbf{d}^{\mathrm{pre}} = \mathbf{G}\mathbf{A}^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{C}_d^{-1}\mathbf{d}^{\mathrm{obs}} + \mathbf{G}\mathbf{A}^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{h}^{\mathrm{pri}} - \mathbf{G}[\mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{H}]^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{h}^{\mathrm{pri}}$$

Manipulate algebraically:

$$\Delta\mathbf{d}^{\mathrm{pre}} = \mathbf{G}\mathbf{A}^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{C}_d^{-1}\mathbf{d}^{\mathrm{obs}} + \mathbf{G}\mathbf{A}^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{h}^{\mathrm{pri}} - \mathbf{G}[\mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{H}]^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{h}^{\mathrm{pri}}$$

$$= \mathbf{G}\mathbf{A}^{-1}\left\{\mathbf{G}^{\mathrm{T}}\mathbf{C}_d^{-1}\mathbf{d}^{\mathrm{obs}} + \mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{h}^{\mathrm{pri}} - \mathbf{A}[\mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{H}]^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{h}^{\mathrm{pri}}\right\}$$

$$= \mathbf{G}\mathbf{A}^{-1}\left\{\mathbf{G}^{\mathrm{T}}\mathbf{C}_d^{-1}\mathbf{d}^{\mathrm{obs}} + \mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{h}^{\mathrm{pri}} - [\mathbf{G}^{\mathrm{T}}\mathbf{C}_d^{-1}\mathbf{G} + \mathbf{H}^{\mathrm{T}}\mathbf{C}_d^{-1}\mathbf{H}][\mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{H}]^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{h}^{\mathrm{pri}}\right\}$$

$$= \mathbf{G}\mathbf{A}^{-1}\left\{\mathbf{G}^{\mathrm{T}}\mathbf{C}_d^{-1}\mathbf{d}^{\mathrm{obs}} + \mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{h}^{\mathrm{pri}} - [\mathbf{G}^{\mathrm{T}}\mathbf{C}_d^{-1}\mathbf{G}][\mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{H}]^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{h}^{\mathrm{pri}} \right.$$
$$\left. - [\mathbf{H}^{\mathrm{T}}\mathbf{C}_d^{-1}\mathbf{H}][\mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{H}]^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{h}^{\mathrm{pri}}\right\}$$

$$= \mathbf{G}\mathbf{A}^{-1}\left\{\mathbf{G}^{\mathrm{T}}\mathbf{C}_d^{-1}\mathbf{d}^{\mathrm{obs}} + \mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{h}^{\mathrm{pri}} - [\mathbf{G}^{\mathrm{T}}\mathbf{C}_d^{-1}\mathbf{G}][\mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{H}]^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{h}^{\mathrm{pri}} - \mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{h}^{\mathrm{pri}}\right\}$$

$$= \mathbf{G}\mathbf{A}^{-1}\left\{\mathbf{G}^{\mathrm{T}}\mathbf{C}_d^{-1}\mathbf{d}^{\mathrm{obs}} - [\mathbf{G}^{\mathrm{T}}\mathbf{C}_d^{-1}\mathbf{G}][\mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{H}]^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{C}_h^{-1}\mathbf{h}^{\mathrm{pri}}\right\}$$

$$= \mathbf{G}\mathbf{A}^{-1}\left\{\mathbf{G}^{\mathrm{T}}\mathbf{C}_d^{-1}\mathbf{d}^{\mathrm{obs}} - [\mathbf{G}^{\mathrm{T}}\mathbf{C}_d^{-1}\mathbf{G}]\mathbf{m}^{(\mathrm{H})}\right\}$$

$$= \mathbf{G}\mathbf{A}^{-1}\left\{\mathbf{G}^{\mathrm{T}}\mathbf{C}_d^{-1}\mathbf{d}^{\mathrm{obs}} - \mathbf{G}^{\mathrm{T}}\mathbf{C}_d^{-1}\mathbf{d}^{(\mathrm{H})}\right\}$$

$$= \mathbf{G}\mathbf{A}^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{C}_d^{-1}\left\{\mathbf{d}^{\mathrm{obs}} - \mathbf{d}^{(\mathrm{H})}\right\}$$

$$= \mathbf{G}\mathbf{A}^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{C}_d^{-1}\left\{\mathbf{d}^{\mathrm{obs}} - \mathbf{d}^{(\mathrm{H})}\right\} = \mathbf{N}\,\Delta\mathbf{d}^{\mathrm{obs}} \quad \text{with} \quad \mathbf{N} \equiv \mathbf{G}\mathbf{A}^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{C}_d^{-1}$$

Q.E.D.