Making surface wave seismograms using the Brune (1962) formula
Bill Menke, April 8, 2021

This is a simple application of the stationary phase approximation:

$$v(X_0, t_0) \cong \frac{A_2(\omega_0)\exp\left[-\frac{\omega_0 t_0}{2Q}\right]}{\sqrt{\sin\theta}\ \sqrt{X\left|\frac{d^2k}{d\omega^2}\right|_0}}\cos(\omega_0 t_0 - k_0 X_0 - \pi/4 - m\pi/2 + \pi/4)$$

Only three quantities need to be specified:

- Source amplitude $A(\omega)$
- Dispersive surface wave phase velocity $c(\omega)$
- Quality factor Q

Here's the Reference:

Bulletin of the Seismological Society of America. Vol. 52, No. 1, pp. 109–112. January, 1962

ATTENUATION OF DISPERSED WAVE TRAINS

By James N. Brune

source amplitude

quality facor

time

$$v(X_0\,,t_0) \cong \frac{A_2(\omega_0)\,\exp\left[-\dfrac{\omega_0\,t_0}{2Q}\right]}{\sqrt{\sin\theta}\;\sqrt{X\left|\dfrac{d^2k}{d\omega^2}\right|_0}}\cos\left(\omega_0\,t_0 - k_0\,X_0 - \pi/4 - m\pi/2 + \pi/4\right)$$

seismogram

distance in km

distance in deg

center frequency

Her, the angular frequency is ω

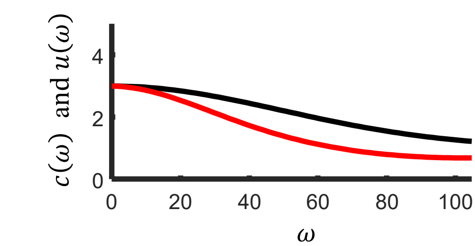
the phase velocity $c(\omega)$ is specified

and these quantities are calculated:

wavenumber $k(\omega) = \frac{\omega}{c(\omega)}$

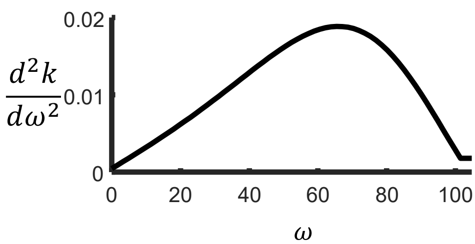
group velocity $u(\omega) = \left(\frac{dk}{d\omega}\right)^{-1}$ (red)

time-dependent center frequency $\omega_0(t)$
the frequency ω_0 for which $u(\omega) = x/t$
(so it depends on x, too)



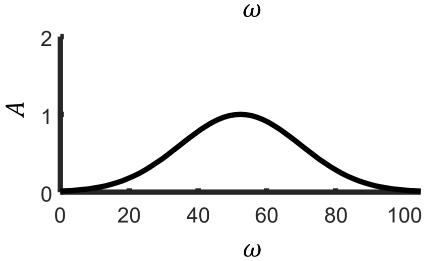
The phase velocity $c(\omega)$ (black) has been specified (it's just a biased Gaussian)

The group velocity $u(\omega)$ (red) is calculated.



$$\frac{d^2k}{d\omega^2}$$

This second derivative of wavenumber is needed because it contributes to surface wave amplitude



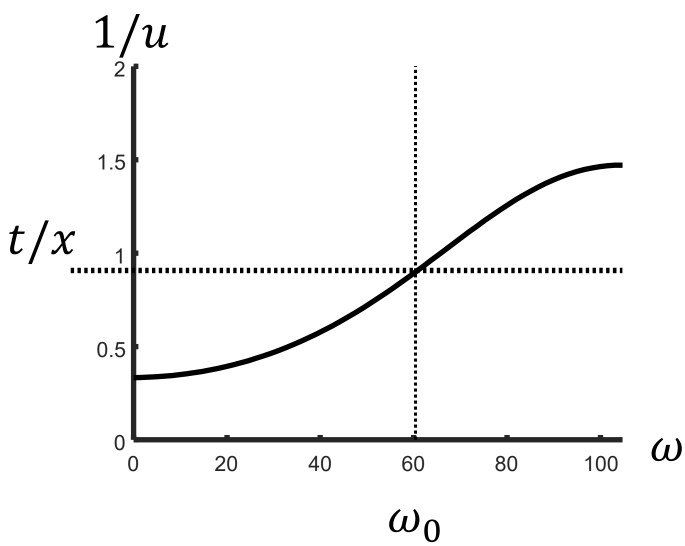
The source amplitude $A(\omega)$ has been specified (it's just a Gaussian).

Procedure to calculate $\omega_0(t)$:

(1) calculate the group slowness $1/u$

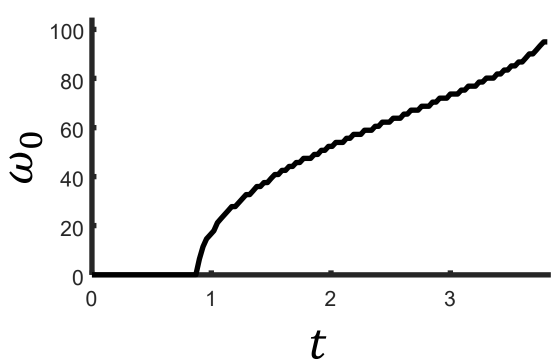
(2) specify a distance x

(3) for each time t in the seismogram look up the frequency ω_0 for which $t/x = 1/\mathrm{u}$. If the equation has no solution, set ω_0=0.



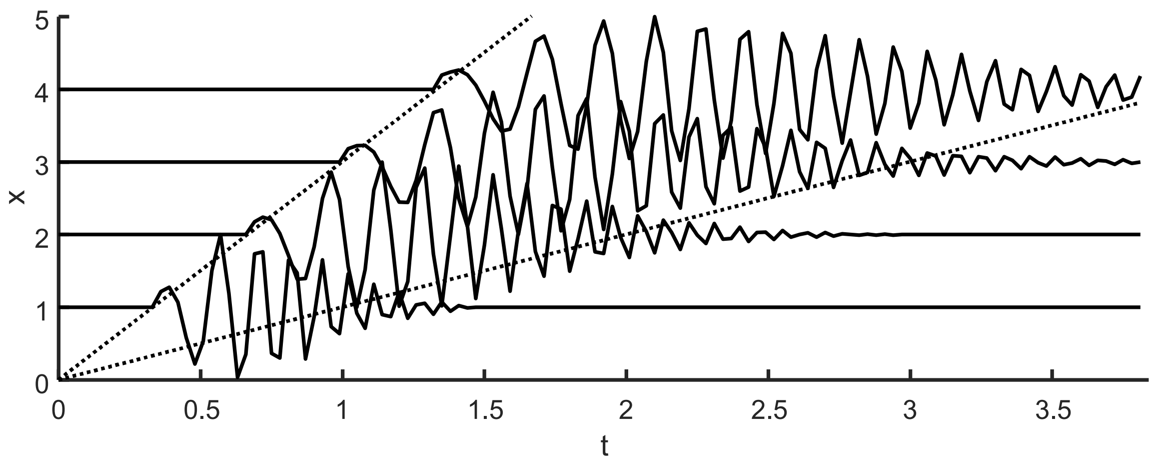
Note: Brune's method fails when $u(\omega)$ has a local maximum or minimum

Here's a plot of $\omega_0(t)$ for an exemplary x



Once $\omega_0(t)$ is known, the Brune (1962) formula can be directly evaluated for the seismogram $v(x,t)$

Here's a sample record section.



```
% setup for time
N=128;
Dt = 0.03;
t = Dt*[0:N-1]';

% setup for Fourier transform
T=N*Dt;
fmax=1/(2*Dt);
wmax = 2*pi*fmax;
Df =fmax/(N/2);
Dw = 2*pi*Df;
Nf=N/2+1;
f=Df*[0:N/2,-N/2+1:-1]';
w = 2*pi*f;
fp = f(1:Nf);
wp = w(1:Nf);
clo = 3;
chi = 1;
sc = 70;

% phase velocity c = w/k
cR = chi +  (clo-chi) * exp( -(wp/sc).^2  );
% wavenumber k = w/c
kR = wp ./ cR;
% group slowness dk/dw
dkdw = diff(kR)/Dw;
dkdw = [dkdw; dkdw(end)];
% group velocity dw/dk
dwdk = 1./dkdw;
uR = dwdk;
kuR = wp ./ uR;
% amplitude factor
d2kdw2 = diff(dkdw)/Dw;
d2kdw2 = [d2kdw2(1:Nf-2); d2kdw2(end-1); d2kdw2(end-1)];
% source wavelet
fc = fmax/2;
sdf = fmax/6;
S = exp( -0.5*((fp-fc).^2)/(sdf^2));
```

```
Q = 30;
xlist = [1, 2, 3, 4];
Nx = length(xlist);
for ix=[1:Nx]
   x=xlist(ix);
   theta = x/111.12;
   sqrtsintheta = sqrt( sin( (pi/180.0)*theta ));
   tox = t/x;
   ws = zeros( N, 1 );
   iws = zeros( N, 1 );
   for i=[1:N]
      Y = tox(i);
      j = find( Y<=dkdw, 1);
      if( isempty(j) )
            continue;
      elseif (j<2)
            continue;
      elseif (j>N)
            continue;
      end
      Y2=dkdw(j);
      Y1=dkdw(j-1);
      X2=wp(j);
      X1=wp(j-1);
      X=X1 + (Y-Y1)*(X2-X1)/(Y2-Y1);
      ws(i) = X;
      iws(i) = floor(0.5+X/Dw) + 1;
   end

   v = zeros(N,1);
   for i=[1:N]
      if( ws(i)<=0 )
         continue;
      end
      j = iws(i);
      A = (S(j)/(2*pi));
      B = sqrt((2*pi)/(x*d2kdw2(j)))/sqrtsintheta;
      C = cos( kR(j)*x - wp(j)*t(i) );
      D = exp( -wp(j)*t(i) / (2*Q) );
      v(i) = A*B*C*D;
   end

   vabsmax = max(abs(v));
   plot( t, x+v/vabsmax, 'k-', 'LineWidth', 2 );
end
```