An Iterated Integral Converted to a Single Integral

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The following formula has application to a problem in ray theory problem that I have been studying:

$$\int_0^s \left[\int_0^{s'} f(s")\, ds" \right] ds' = \int_0^s [s - s'] f(s')\, ds'$$

Perhaps it is well known, but today was the first time I've encountered it.

I first approached the problem by approximating the integral as a sum

$$J(s) \equiv \int_0^s \left[\int_0^{s'} f(s")\, ds" \right] ds' \equiv \int_0^s I(s')\, ds' \quad \text{with} \quad I(s') \equiv \int_0^{s'} f(s")\, ds"$$

$$I(s') = \int_0^{s'} f(s")\, ds" \approx \Delta s \sum_{n=1}^{N} f(s_n) = \Delta s \sum_{n=1}^{N} f_n \quad \text{with} \quad f_n \equiv f(s_n) \quad \text{and} \quad s_n \equiv n\Delta s$$

Then, for each value of s_n, I compute the cumulative sum yielding $I(n\Delta s)$

$I(\Delta s) \approx \Delta s\{f_1\}$

$I(2\Delta s) \approx \Delta s\{f_1 + f_2\}$

$I(3\Delta s) \approx \Delta s\{f_1 + f_2 + f_3\}$

$I(N\Delta s) \approx \Delta s\{f_1 + f_2 + f_3 + \cdots + f_N\}$

and then take the cumulative sum of I's to get $J(n\Delta s)$

$J(\Delta s) \approx (\Delta s)^2\{f_1\}$

$J(2\Delta s) \approx (\Delta s)^2\{2f_1 + f_2\}$

$J(3\Delta s) \approx (\Delta s)^2\{3f_1 + 2f_2 + f_3\}$

$J(N\Delta s) \approx (\Delta s)^2\{(N)f_1 + (N-1)f_2 + (N-2)f_3 + \cdots + (1)f_N\}$

Then $J(N\Delta s)$ can be written:

$$J(N\Delta s) = \Delta s\{s_N f_1 + s_{N-1} f_2 + s_{N-2} f_3 + \cdots + s_1 f_N\} = \Delta s \sum_{n=1}^{N} s_{N-n+1} f_n$$

or since $s_{(N+1)-n} = s_{N+1} - s_n$ with

$$J(s_N) \approx \Delta s \sum_{n=1}^{N} (s_{N+1} - s_n)\, f_n$$

it must be that

$$J(s_{max}) = \int_0^{s_{max}} (s_{max} - s)\, f(s)\, ds$$

It took me a while to figure out how to prove this formula. The trick is to introduce the Heaviside step function $H(s_2 - s_1)$, defined to be unity when $s_2 > s_1$ and zero otherwise. This allows the top limit in the inner integral to be increased to s:

$$J(s) \equiv \int_0^s \left[\int_0^{s'} f(s")\, ds"\right] ds' = \int_0^s \left[\int_0^{s} H(s' - s")f(s")\, ds"\right] ds'$$

Then, the order of the arguments in the Heaviside function are reversed, using the identity $H(s' - s") = 1 - H(s" - s')$.

$$J(s) = \int_0^s \left[\int_0^{s} \{1 - H(s" - s')\} f(s")\, ds"\right] ds'$$

Then, the order of integration is swapped:

$$J(s) = \int_0^s \left[\int_0^{s} \{1 - H(s" - s')\}\, ds'\right] f(s")\, ds"$$

The inner integral can now be performed analytically, yielding

$$J(s) = \int_0^s [s - s"] f(s")\, ds"$$

Note, however that the integrand depends upon the upper limit of the integral. For numerical calculations, that means that the formula is only really useful if one wants to know $J(s)$ at an endpoint (as contrasted at all points between zero and the endpoint).

Here’s a numerical demonstration of the result.

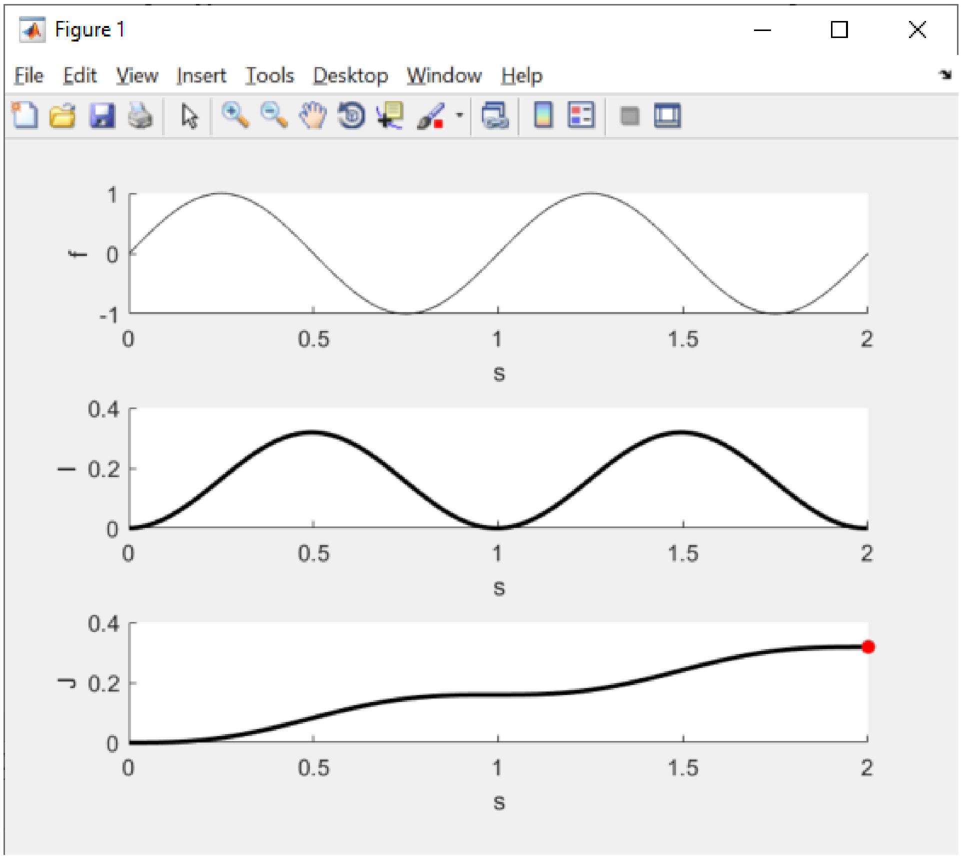

Figure 1. Test with a sinusoidal $f(s)$. (Top) the function $f(s)$. (Middle) The function $I(s)$, computed via numerical integration. (Bottom) The function $J(s)$, computed via numerical iterated integration, with the results of the formula, evaluated for the endpoint, shown with a red dot.