Scattered P-waves Fully Characterize a Point Heterogeneity

Bill Menke, May 26, 2021

In summary, the functions $f_1(\theta)$ and $f_2(\theta)$, observed on the interval $0 \le \theta < 2\pi$, are sufficient to uniquely characterize the point heterogeneity, by determining $\delta\rho/\rho$, $\delta\beta/\beta$, $\delta\alpha/\alpha$, and as a bonus, give the velocity ratio β/α of the background medium

Rondenay (2009, DOI 10.1007/s10712-009-9071-5, Equations 14-15) gives the scattering functions for P wave interacting with a point heterogeneity as:

$$f^{P\to P}(\mathbf{x},\theta) = \rho\left(2\frac{\delta\alpha}{\alpha} + \frac{\delta\beta}{\beta}\left(2\left(\frac{\beta}{\alpha}\right)^2(\cos 2\theta - 1)\right) + \frac{\delta\rho}{\rho}\left(1 + \cos\theta + \left(\frac{\beta}{\alpha}\right)^2(\cos 2\theta - 1)\right)\right), \tag{14}$$

$$f^{P\to S}(\mathbf{x},\theta) = \rho\left(\frac{\delta\beta}{\beta}\left(2\frac{\beta}{\alpha}\sin 2\theta\right) + \frac{\delta\rho}{\rho}\left(\sin\theta + \frac{\beta}{\alpha}\sin 2\theta\right)\right), \tag{15}$$

These functions are plotted in Figure 1. The equations can be rearranged:

$$\begin{bmatrix} f_1(\theta) \\ f_2(\theta) \end{bmatrix} \equiv \begin{bmatrix} \rho f^{P\to P} \\ \rho f^{P\to S} \end{bmatrix} = \begin{bmatrix} 2 & \{2r^2(\cos(2\theta)-1)\} & \{1+\cos(\theta)+r^2(\cos(2\theta)-1)\} \\ 0 & 2r\sin(2\theta) & \{\sin(\theta)+r\sin(2\theta)\} \end{bmatrix} \begin{bmatrix} \delta\alpha/\alpha \\ \delta\beta/\beta \\ \delta\rho/\rho \end{bmatrix}$$

with $r = \beta/\alpha$. Reorganizing this equation with trigonometric functions as right-hand vector:

$$\begin{bmatrix} f_1(\theta) \\ f_2(\theta) \end{bmatrix} =$$

$$\begin{bmatrix} \left(2\frac{\delta\alpha}{\alpha} - 2r^2\frac{\delta\beta}{\beta} + (1-r^2)\frac{\delta\rho}{\rho}\right) & \left(\frac{\delta\rho}{\rho}\right) & 0 & \left(2r^2\frac{\delta\beta}{\beta} + r^2\frac{\delta\rho}{\rho}\right) & 0 \\ 0 & 0 & \left(\frac{\delta\rho}{\rho}\right) & 0 & \left(2r\frac{\delta\beta}{\beta} + r\frac{\delta\rho}{\rho}\right) \end{bmatrix} \begin{bmatrix} 1 \\ \cos(\theta) \\ \sin(\theta) \\ \cos(2\theta) \\ \sin(2\theta) \end{bmatrix}$$

$$\equiv \mathbf{Mg}$$

Here the angular functions $g_n(\theta), n = 1 \dots 5$ are given by:

$$\begin{bmatrix} g_1(\theta) \\ g_2(\theta) \\ g_3(\theta) \\ g_4(\theta) \\ g_5(\theta) \end{bmatrix} \equiv \begin{bmatrix} 1 \\ \cos(\theta) \\ \sin(\theta) \\ \cos(2\theta) \\ \sin(2\theta) \end{bmatrix}$$

It can be shown that the angular functions are orthogonal:

$$\int_0^{2\pi} g_n(\theta)\, g_m(\theta)\, \mathrm{d}\theta = N_n^2 \delta_{nm}$$

Here N_n^2 is a normalization factor. Applying the integral to the equation yields:

$$\begin{bmatrix} d_{1n} \\ d_{2n} \end{bmatrix} \equiv N_n^{-2} \begin{bmatrix} \int_0^{2\pi} f_1(\theta)\, g_n(\theta)\, \mathrm{d}\theta \\ \int_0^{2\pi} f_2(\theta)\, g_n(\theta)\, \mathrm{d}\theta \end{bmatrix} = N_n^{-2} \sum_{m=1}^{5} \begin{bmatrix} M_{1m} \\ M_{2m} \end{bmatrix} \int_0^{2\pi} g_n(\theta) g_m(\theta)\, \mathrm{d}\theta = \sum_{m=1}^{5} \begin{bmatrix} M_{1m} \\ M_{2m} \end{bmatrix} \delta_{nm} = \begin{bmatrix} M_{1n} \\ M_{2n} \end{bmatrix}$$

The non-zero elements provide five equations linking $\frac{\delta\alpha}{\alpha}$, $\frac{\delta\beta}{\beta}$ and $\frac{\delta\rho}{\rho}$ to the ds:

$$\left(2\frac{\delta\alpha}{\alpha} - 2r^2\frac{\delta\beta}{\beta} + (1 - r^2)\frac{\delta\rho}{\rho}\right) = d_{11}$$

$$\left(\frac{\delta\rho}{\rho}\right) = d_{12}$$

$$\left(2r^2\frac{\delta\beta}{\beta} + r^2\frac{\delta\rho}{\rho}\right) = d_{14}$$

$$\left(\frac{\delta\rho}{\rho}\right) = d_{23}$$

$$\left(2r\frac{\delta\beta}{\beta} + r\frac{\delta\rho}{\rho}\right) = d_{25}$$

The five equations are solved as follows: From equations 2 and 4:

$$\left(\frac{\delta\rho}{\rho}\right) = d_{12} = d_{23}$$

If r is presumed known, then from equations 3 and 5:

$$\left(\frac{\delta\beta}{\beta}\right) = \frac{d_{14} - r^2 d_{12}}{2r^2} = \frac{d_{25} - r d_{12}}{2r}$$

Alternatively, when r is presumed unknown, from equations 3 and 5

$$r^2\left(2\frac{\delta\beta}{\beta} + d_{12}\right) = d_{14} \quad \text{so} \quad r^2 = \frac{d_{14}}{\left(2\frac{\delta\beta}{\beta} + d_{12}\right)}$$

$$r\left(2\frac{\delta\beta}{\beta} + d_{12}\right) = d_{25} \quad \text{so} \quad r^2 = \frac{(d_{25})^2}{\left(2\frac{\delta\beta}{\beta} + d_{12}\right)^2}$$

$$\frac{d_{14}}{\left(2\frac{\delta\beta}{\beta}+d_{12}\right)}=\frac{(d_{25})^2}{\left(2\frac{\delta\beta}{\beta}+d_{12}\right)^2}$$

$$d_{14}\left(2\frac{\delta\beta}{\beta}+d_{12}\right)^2=(d_{25})^2\left(2\frac{\delta\beta}{\beta}+d_{12}\right)$$

$$4d_{14}\left(\frac{\delta\beta}{\beta}\right)^2+(4d_{12}d_{14}-2(d_{25})^2)\left(\frac{\delta\beta}{\beta}\right)+(d_{14}(d_{12})^2-(d_{25})^2d_{12})=0$$

$$A\equiv 4d_{14}\quad\text{and}\quad B\equiv 4d_{12}d_{14}-2(d_{25})^2\quad\text{and}\quad C\equiv d_{14}(d_{12})^2-(d_{25})^2d_{12}$$

$$B^2-4A=16(d_{12})^2(d_{14})^2+4(d_{25})^4-\ 16d_{12}d_{14}(d_{25})^2-16(d_{12})^2(d_{14})^2 \\ +16d_{12}d_{14}(d_{25})^2=4(d_{25})^4$$

$$\sqrt{B^2-4A}=2(d_{25})^2$$

$$\left(\frac{\delta\beta}{\beta}\right)=\frac{-B\pm\sqrt{B^2-4AC}}{2A}=\frac{-4d_{12}d_{14}+2(d_{25})^2\ \pm 2(d_{25})^2}{8d_{14}}$$

Note that for the $-$ of the $\pm$, $\left(\frac{\delta\beta}{\beta}\right)=-½d_{12}$. We will show that this root is not allowed, since it leads to a singular r. Consequently, the correct root is the $+$ case:

$$\left(\frac{\delta\beta}{\beta}\right)=\frac{(d_{25})^2\ -d_{12}d_{14}}{2d_{14}}$$

Then from equation 5:

$$r=\frac{d_{25}}{\left(2\frac{\delta\beta}{\beta}+d_{12}\right)}$$

As anticipated, r is singular when $\frac{\delta\beta}{\beta}=-½d_{12}$. Finally, from equation 1:

$$\left(\frac{\delta\alpha}{\alpha}\right)=½d_{11}+r^2\frac{\delta\beta}{\beta}-½(1-r^2)\frac{\delta\rho}{\rho}$$

Finally, note that the $P\rightarrow P$ interaction (equations 1-3) are sufficient to solve for $\delta\alpha/\alpha$, $\delta\beta/\beta$ and $\delta\rho/\rho$ but not r, and that the $P\rightarrow S$ interaction (equations 4-5) are sufficient to solve for $\delta\beta/\beta$ and $\delta\rho/\rho$ but not $\delta\alpha/\alpha$ or r.

In summary, the functions $f_1(\theta)$ and $f_2(\theta)$, observed on the interval $0\leq\theta<2\pi$, are sufficient to uniquely characterize the point heterogeneity, by determining $\delta\rho/\rho$, $\delta\beta/\beta$, $\delta\alpha/\alpha$, and as a bonus, give the velocity ratio β/α of the background medium.

Note: I have checked the algebra numerically:

Output of a MATLAB script that checks the algebra:

```
>> Eqn14
error in eqn 1:  0.0000
error in eqn 2:  0.0000
error in eqn 3:  0.0000
error in eqn 4:  0.0000
error in eqn 5:  -0.0000
dror true 0.1100 est1 0.1100 est2 0.1100
dbob true 1.0700 est1 1.0700 est2 1.0700 (with correct r)
two versions of B^2-4C: 11.3906 11.3906
r denominator 2.2500 0.0000
dbob true 1.0700 est 1.0700 (with variable r)
r true 0.5774 est 0.5774
daoa true 0.5100 est 0.5100
Total L1 error 0.0000
```

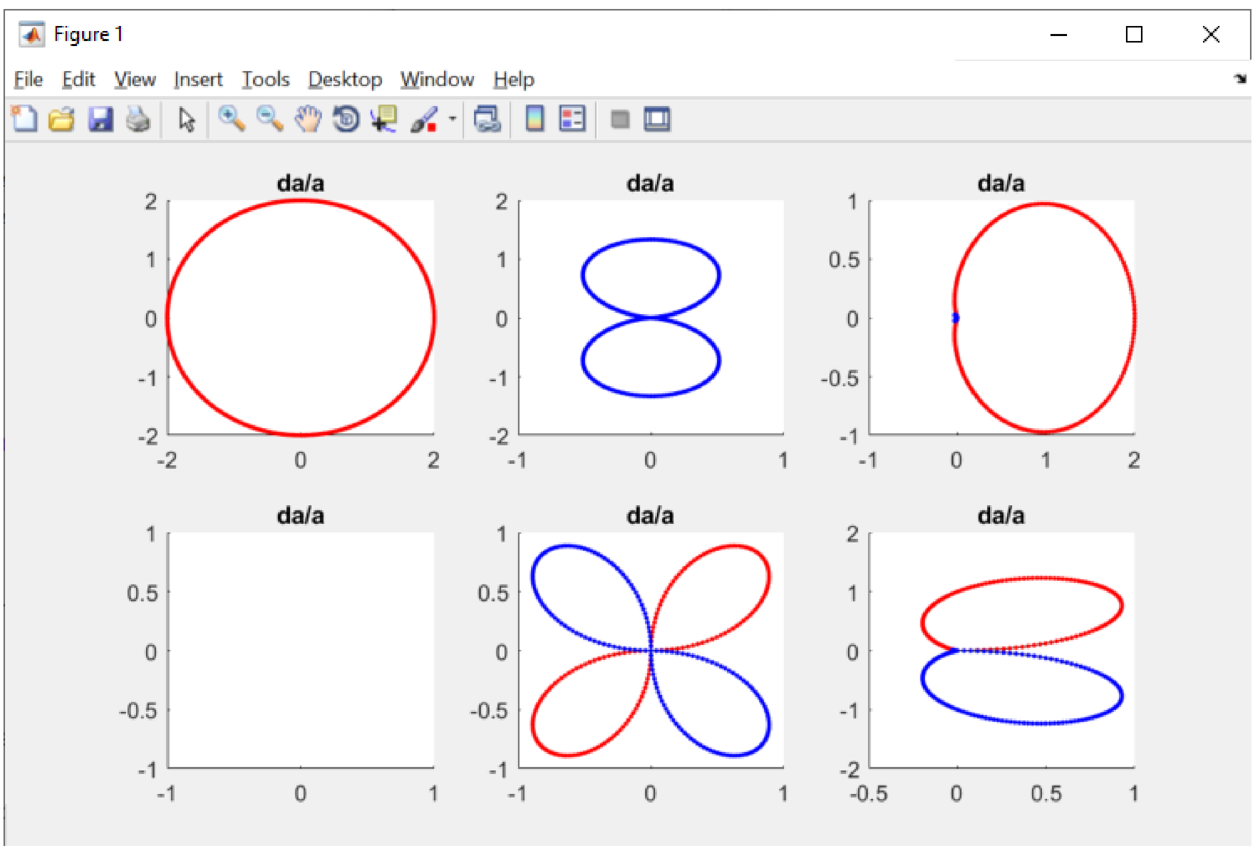

Fig. 1. (Top row) The function $f_1(\theta)$ when (left) only $\delta\alpha/\alpha \neq 0$, (middle) only $\delta\beta/\beta \neq 0$; and (right) only $\delta\rho/\rho \neq 0$. (Bottom row) Same as top row, except for $f_2(\theta)$. Incident P wave propagates from left to right. Positive fs are shown in red, negative in blue.