Forwarded Scattered P-waves Fully Characterize a Point Heterogeneity

Bill Menke, June 3, 2021

This is a quick modification to the previous note. Here I consider forward scattering, only.

Summary: the $P \rightarrow P$ forward scattering is sufficient to solve for $\delta\alpha/\alpha$, $\delta\beta/\beta$ and $\delta\rho/\rho$ and the $P \rightarrow S$ forward scattering is sufficient to solve for $\delta\beta/\beta$ and $\delta\rho/\rho$. When both are measured, one can also solve for $r = \beta/\alpha$.

Rondenay (2009, DOI 10.1007/s10712-009-9071-5, Equations 14-15) gives the scattering functions for P wave interacting with a point heterogeneity as $\mathbf{f} = \mathbf{Hh}$:

$$\begin{bmatrix} f_1(\theta) \\ f_2(\theta) \end{bmatrix} =$$

$$\begin{bmatrix} \left(2\frac{\delta\alpha}{\alpha} - 2r^2\frac{\delta\beta}{\beta} + (1-r^2)\frac{\delta\rho}{\rho}\right) & \left(\frac{\delta\rho}{\rho}\right) & 0 & \left(2r^2\frac{\delta\beta}{\beta} + r^2\frac{\delta\rho}{\rho}\right) & 0 \\ 0 & 0 & \left(\frac{\delta\rho}{\rho}\right) & 0 & \left(2r\frac{\delta\beta}{\beta} + r\frac{\delta\rho}{\rho}\right) \end{bmatrix} \begin{bmatrix} 1 \\ \cos(\theta) \\ \sin(\theta) \\ \cos(2\theta) \\ \sin(2\theta) \end{bmatrix}$$

with $r = \beta/\alpha$. For forward-scattered waves $-\pi/2 < \theta < +\pi/2$. The above angular functions $\mathbf{h}$ are not orthogonal on this interval, but the functions $g_n(\theta), n = 1 \ldots 5$ are:

$$\begin{bmatrix} g_1(\theta) \\ g_2(\theta) \\ g_3(\theta) \\ g_4(\theta) \\ g_5(\theta) \end{bmatrix} \equiv \begin{bmatrix} 1 \\ \cos(\theta) - c_1 - c_4\cos(2\theta) \\ \sin(\theta) - c_3\ \sin(2\theta) \\ \cos(2\theta) \\ \sin(2\theta) \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} c_1 \\ c_4 \\ c_3 \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{\pi}\right) \\ \left(\frac{4}{3\pi}\right) \\ \left(\frac{8}{3\pi}\right) \end{bmatrix}$$

By orthogonal, I mean $\int_0^{2\pi} g_n(\theta)\, g_m(\theta)\, \mathrm{d}\theta = N_n^2 \delta_{nm}$ with N_n a normalization constant. I have verified the orthogonality numerically. We find the elements of $\mathbf{G}$ by writing $\mathbf{f} = \mathbf{Gg}$ and equating coefficients in $\mathbf{Hh} = \mathbf{Gg}$:

first row:

$$f_1 = H_{11}h_1 + H_{12}h_2 + H_{14}h_4 = G_{11}g_1 + G_{12}g_2 + G_{14}g_4 =$$

$$= G_{11}h_1 + G_{12}(h_2 - c_1h_1 - c_4h_4) + G_{14}h_4 =$$

$$= G_{12}h_2 + (G_{11} - G_{12}c_1)h_1 + (G_{14} - G_{12}c_4)$$

$$G_{12} = H_{12} \quad \text{and} \quad H_{11} = G_{11} - G_{12}c_1 \quad \text{and} \quad H_{14} = G_{14} - G_{12}c_4$$

$$G_{12} = H_{12} \quad \text{and} \quad G_{11} = H_{11} + H_{12}c_1 \quad \text{and} \quad G_{14} = H_{14} + H_{12}c_4$$

second row:

$$f_2 = H_{23}h_3 + H_{25}h_5 = G_{23}g_3 + G_{25}g_5 = G_{23}(h_3 - c_3h_5) + G_{25}h_5$$
$$= G_{23}h_3 + (G_{25} - G_{23}c_3)h_5$$

$$G_{23} = H_{23} \quad \text{and} \quad H_{25} = G_{25} - G_{23}c_3 \quad \text{and} \quad G_{25} = H_{25} + H_{23}c_3$$

We now write the non-zero components of **G** as:

$$G_{11} = H_{11} + H_{12}c_1 = 2\left(\frac{\delta\alpha}{\alpha}\right) - 2r^2\left(\frac{\delta\beta}{\beta}\right) + (1 - r^2)\left(\frac{\delta\rho}{\rho}\right) + \left(\frac{2}{\pi}\right)\left(\frac{\delta\rho}{\rho}\right) =$$

$$= 2\left(\frac{\delta\alpha}{\alpha}\right) - 2r^2\left(\frac{\delta\beta}{\beta}\right) + \left(1 + \left(\frac{2}{\pi}\right) - r^2\right)\left(\frac{\delta\rho}{\rho}\right)$$

$$G_{12} = H_{12} = \left(\frac{\delta\rho}{\rho}\right)$$

$$G_{14} = H_{14} + H_{12}c_4 = 2r^2\left(\frac{\delta\beta}{\beta}\right) + r^2\left(\frac{\delta\rho}{\rho}\right) + \left(\frac{4}{3\pi}\right)\left(\frac{\delta\rho}{\rho}\right) =$$

$$= 2r^2\left(\frac{\delta\beta}{\beta}\right) + \left(r^2 + \left(\frac{4}{3\pi}\right)\right)\left(\frac{\delta\rho}{\rho}\right)$$

$$G_{23} = H_{23} = \left(\frac{\delta\rho}{\rho}\right)$$

$$G_{25} = H_{25} + H_{23}c_3 = 2r\left(\frac{\delta\beta}{\beta}\right) + r\left(\frac{\delta\rho}{\rho}\right) + \left(\frac{8}{3\pi}\right)\left(\frac{\delta\rho}{\rho}\right) =$$

$$= 2r\left(\frac{\delta\beta}{\beta}\right) + \left(r + \left(\frac{8}{3\pi}\right)\right)\left(\frac{\delta\rho}{\rho}\right)$$

Applying the integral to the equation $\mathbf{f} = \mathbf{Gg}$ yields:

$$\begin{bmatrix} d_{1n} \\ d_{2n} \end{bmatrix} \equiv N_n^{-2}\begin{bmatrix} \int_0^{2\pi} f_1(\theta)\, g_n(\theta)\, \mathrm{d}\theta \\ \int_0^{2\pi} f_2(\theta)\, g_n(\theta)\, \mathrm{d}\theta \end{bmatrix} = N_n^{-2}\sum_{m=1}^{5}\begin{bmatrix} G_{1m} \\ G_{2m} \end{bmatrix}\int_0^{2\pi} g_n(\theta)g_m(\theta)\, \mathrm{d}\theta = \sum_{m=1}^{5}\begin{bmatrix} G_{1m} \\ G_{2m} \end{bmatrix}\delta_{nm} = \begin{bmatrix} G_{1n} \\ G_{2n} \end{bmatrix}$$

The non-zero elements provide five equations linking $\frac{\delta\alpha}{\alpha}$, $\frac{\delta\beta}{\beta}$ and $\frac{\delta\rho}{\rho}$ to the ds:

$$2\left(\frac{\delta\alpha}{\alpha}\right) - 2r^2\left(\frac{\delta\beta}{\beta}\right) + \left(1 + \left(\frac{2}{\pi}\right) - r^2\right)\left(\frac{\delta\rho}{\rho}\right) = d_{11}$$

$$\left(\frac{\delta\rho}{\rho}\right) = d_{12}$$

$$2r^2\left(\frac{\delta\beta}{\beta}\right)+\left(r^2+\left(\frac{4}{3\pi}\right)\right)\left(\frac{\delta\rho}{\rho}\right)=d_{14}$$

$$\left(\frac{\delta\rho}{\rho}\right)=d_{23}$$

$$2r\left(\frac{\delta\beta}{\beta}\right)+\left(r+\left(\frac{8}{3\pi}\right)\right)\left(\frac{\delta\rho}{\rho}\right)=d_{25}$$

Let's assume that r is known. Then, the top three equations give:

$$\left(\frac{\delta\rho}{\rho}\right)=d_{12}$$

$$\left(\frac{\delta\beta}{\beta}\right)=\frac{d_{14}-\left(r^2+\left(\frac{4}{3\pi}\right)\right)d_{12}}{2r^2}$$

$$\left(\frac{\delta\alpha}{\alpha}\right)=\frac{d_{11}+2r^2\left(\frac{\delta\beta}{\beta}\right)-\left(1+\left(\frac{2}{\pi}\right)-r^2\right)d_{12}}{2}$$

And from the bottom two equations:

$$\left(\frac{\delta\rho}{\rho}\right)=d_{23}$$

$$\left(\frac{\delta\beta}{\beta}\right)=\frac{d_{25}-\left(r+\left(\frac{8}{3\pi}\right)\right)d_{23}}{2r}$$

I have checked these five equations numerically.

So, the $P\rightarrow P$ interaction (equations 1-3) are sufficient to solve for $\delta\alpha/\alpha$, $\delta\beta/\beta$ and $\delta\rho/\rho$ and that the $P\rightarrow S$ interaction (equations 4-5) are sufficient to solve for $\delta\beta/\beta$ and $\delta\rho/\rho$. In order to solve for r, one must eliminate it from equations 3 and 5, and then solve the resulting equation for $\delta\beta/\beta$:

$$r^2=\frac{d_{14}-\left(\frac{4}{3\pi}\right)d_{12}}{2\left(\frac{\delta\beta}{\beta}\right)+d_{12}}$$

$$r=\frac{d_{25}-\left(\frac{8}{3\pi}\right)d_{12}}{2\left(\frac{\delta\beta}{\beta}\right)+d_{12}}$$

Eliminating r yields a quadratic equation for $\delta\beta/\beta$:

$$\frac{d_{14}-\left(\frac{4}{3\pi}\right)d_{12}}{2\left(\frac{\delta\beta}{\beta}\right)+d_{12}}=\frac{\left(d_{25}-\left(\frac{8}{3\pi}\right)d_{12}\right)^2}{\left(2\left(\frac{\delta\beta}{\beta}\right)+d_{12}\right)^2}$$

$$\left(d_{14}-\left(\frac{4}{3\pi}\right)d_{12}\right)\left(2\left(\frac{\delta\beta}{\beta}\right)+d_{12}\right)^2=\left(2\left(\frac{\delta\beta}{\beta}\right)+d_{12}\right)\left(d_{25}-\left(\frac{8}{3\pi}\right)d_{12}\right)^2$$

$$\left(2\left(\frac{\delta\beta}{\beta}\right)+d_{12}\right)^2=c\left(2\left(\frac{\delta\beta}{\beta}\right)+d_{12}\right)\quad \text{with } c=\frac{\left(d_{25}-\left(\frac{8}{3\pi}\right)d_{12}\right)^2}{\left(d_{14}-\left(\frac{4}{3\pi}\right)d_{12}\right)}$$

$$4\left(\frac{\delta\beta}{\beta}\right)^2+(4d_{12}-2c)\left(\frac{\delta\beta}{\beta}\right)+((d_{12})^2-cd_{12})=0$$

$$so\ A=4 \ \text{ and }\ B=(4d_{12}-2c)\ \text{ and }\ C=((d_{12})^2-cd_{12})$$

The discriminant, $D^2=B^2-4AC$ is

$$D^2=16(d_{12})^2+4c^2-16cd_{12}-16(d_{12})^2+16cd_{12}=4c^2 \quad \text{and} \quad D=2c$$

and the solution is

$$\left(\frac{\delta\beta}{\beta}\right)=\frac{-4d_{12}+2c\pm 2c}{8}$$

The $-d_{12}/2$ solution is unphysical, because it leads to a singular value of r when inserted into the original equations. Hence:

$$\left(\frac{\delta\beta}{\beta}\right)=\frac{c-d_{12}}{2}$$

The quantity $\delta\alpha/\alpha$ can then be determined using the first equation. Thus, when both P and S wave forward scattering is measured, the data can also determine r.

I've not checked the last half of this note.