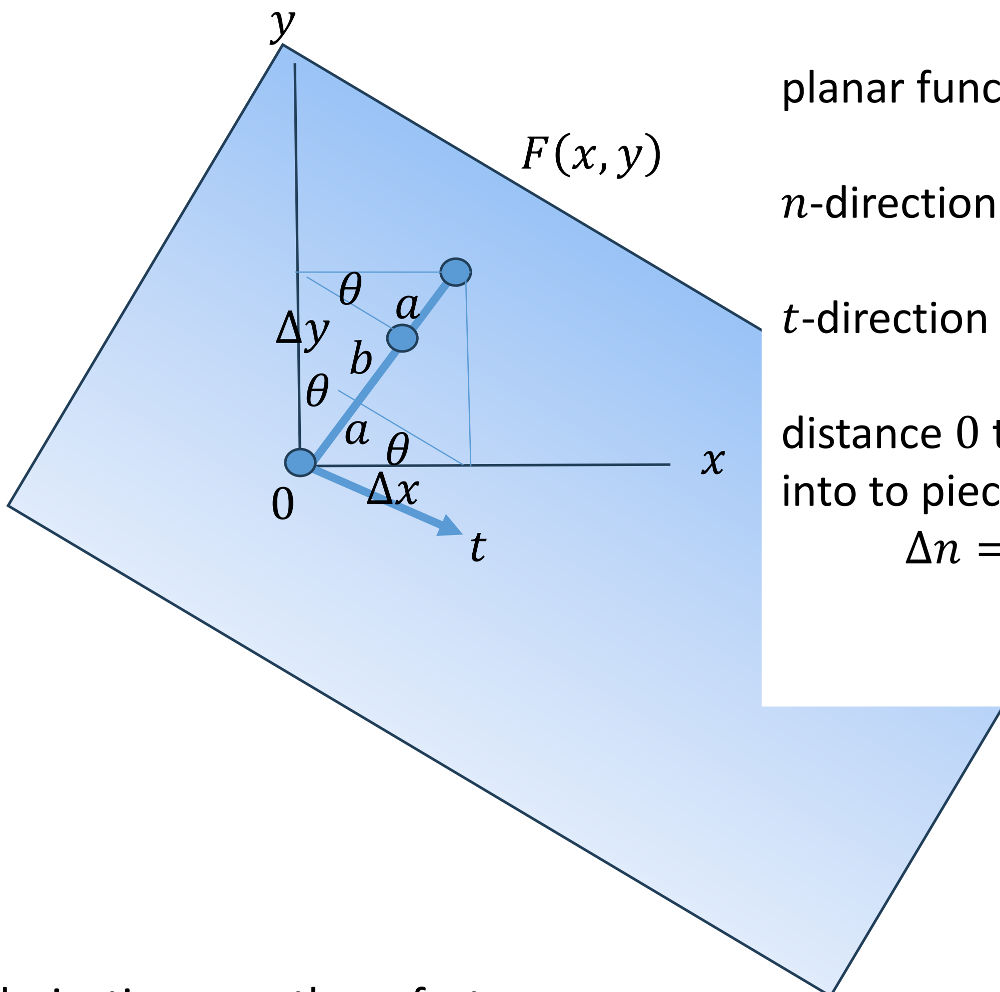
Steepest Slope of a Plane by Measuring Slopes in Two Orthogonal Directions

This derivation intentionally uses no calculus

Bill Menke, 7/11/23



planar function $F(x,y)$

n-direction is up-slope

t-direction is cross-slope

distance 0 to Δn broken into to pieces, a, b, so

$$\Delta n = a + b$$

derivation uses these facts:

movement along t-direction does not change $F(x,y)$

top triangle is same as bottom, as they are right triangles with the same hypotenuse Δx and the same angle θ

change in F from 0 to Δn broken into two parts,
0 to b (of length b) and b to Δn (of length a)

Part 1. relation between polar angle θ and Δx, Δy

$$\tan \theta = \frac{\Delta x}{\Delta y}$$

relation between a and b and $\theta, \Delta x$, Δy

$$\sin \theta = \frac{a}{\Delta x} \quad \cos \theta = \frac{b}{\Delta y} \qquad a = \Delta x \sin \theta \qquad \mathrm{b} = \Delta y \cos \theta$$

Part 2. calculate $\Delta n / \Delta x$ and $\Delta n / \Delta y$

Δn is distance from 0 to a and from a to Δn

$$\Delta n = a + b = \Delta x \sin \theta + \Delta y \cos \theta$$

$$\frac{\Delta n}{\Delta x} = \frac{\Delta x \sin \theta + \Delta y \cos \theta}{\Delta x} = \sin \theta + \frac{\Delta y}{\Delta x} \cos \theta = \sin \theta + \frac{\cos \theta}{\tan \theta}$$

$$= \frac{\sin^2 \theta}{\sin \theta} + \frac{\cos^2 \theta}{\sin \theta} = \frac{1}{\sin \theta}$$

similarly $\dfrac{\Delta n}{\Delta y} = \dfrac{1}{\cos \theta}$

Part 3. relate polar angle θ to $\Delta F/\Delta x$ and $\Delta F/\Delta y$

$$\tan\theta = \frac{\Delta x}{\Delta y} = \frac{\Delta x/\Delta F}{\Delta y/\Delta F} = \frac{(\Delta F/\Delta y)/c}{(\Delta F/\Delta x)/c} \qquad \text{with } c \text{ unknown}$$

$$\sin\theta = (\Delta F/\Delta y)/c \qquad \cos\theta = (\Delta F/\Delta x)/c$$

c determined by condition

$\sin^2\theta + \cos^2\theta = 1$ implies $c^2 = (\Delta F/\Delta x)^2 + (\Delta F/\Delta y)^2$

change in F from 0 to Δn broken into two pieces

0 to b, a distance b

same as $\frac{\Delta F}{\Delta y}\Delta y$

a to Δn, a distance b

same as $\frac{\Delta F}{\Delta x}\Delta x$

so

$$\Delta F = \frac{\Delta F}{\Delta x}\Delta x + \frac{\Delta F}{\Delta y}\Delta y$$

divide by Δn and apply Part 2

$$\frac{\Delta F}{\Delta n} = \frac{\Delta F}{\Delta x}\frac{\Delta x}{\Delta n} + \frac{\Delta F}{\Delta y}\frac{\Delta y}{\Delta n} = \frac{dF}{dx}\sin\theta + \frac{dF}{dy}\cos\theta$$

Part 4. Apply Part 3

$$\frac{\Delta F}{\Delta n} = \frac{dF}{dx}\sin\theta + \frac{dF}{dy}\cos\theta = \frac{1}{c}\left(\frac{dF}{dx}\right)^2 + \frac{1}{c}\left(\frac{dF}{dy}\right)^2 = \frac{c^2}{c} = c$$

$$\frac{\Delta F}{\Delta n} = [(\Delta F/\Delta x)^2 + (\Delta F/\Delta y)^2]^{½}$$

Part 5. write result in terms of slope angles, φ

$$\tan\varphi = \frac{\Delta F}{\Delta n} \quad \tan\varphi_x = \frac{\Delta F}{\Delta x} \quad \tan\varphi_y = \frac{\Delta F}{\Delta y}$$

$$\tan\varphi = \left[(\tan\varphi_x)^2 + \left(\tan\varphi_y\right)^2\right]^{½}$$