In a Horizontally-Propagating Wave, Energy Flux is Energy Density Times Velocity

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Consider a fluid containing a solute (dissolved substance), when the fluid is moving in the x-direction. The amount of solute in the fluid is quantified by its concentration, E, measured in units of kg/m^3. The flux of the solute through the (y, z) plane is e_x, measured in units of kg/m^{2}s. Now suppose that fluid motion is the only mechanism for moving the solute, with the horizontal velocity v_x is measured in units of m/s. Velocity can also be considered to be the volume flux of fluid, measured in units of m^3/m^{2}s. Then, the flux of solute is the volume flux of water times the concentration of solute in that fluid; that is, $e_x = v_x E$, again measured in usint of (kg/m^3)(m^3/m^{2}s) = kg/ m^{2}s.

This scenario is not quite analogous to energy transport in a seismic wave, because while the energy is contained within the rock in the same way that the solute is dissolved in the fluid, the rock is not moving in the same sense that the fluid is. Nevertheless, as we show below, the relationship can be shown to be true.

We assume harmonic motion of the form $\exp(ikx - i\omega t)$, where k is horizontal wavenumber and ω is angular frequency. This wave moved in the positive x-direction at velocity $v_H = \omega/k$. Derivatives of a quantity, say u, with respect to horizontal position x and time t can be performed trivially, as $\dot{u}_{,x} = iku_i$ and $\dot{u}_i = -i\omega u_i$.

Asuming that all motions and stresses are confined to the (x, z) plane, the horizontal energy flux $e_x = -\tau_{xj}\dot{u}_j$ (Synge, 1956-1957) reduces to

$$e_x = i\omega[\tau_{xx}u_x + \tau_{xz}u_z] \quad \text{so} \quad [\tau_{xx}u_x + \tau_{xz}u_z] = \frac{e_x}{i\omega} \tag{A.1}$$

As the wave is propagating horizontally, its vertical energy flux $e_z = -\tau_{zj}\dot{u}_j$ is zero, and we can write

$$e_z = 0 = i\omega[\tau_{xz}u_x + \tau_{zz}u_z] \quad \text{so} \quad [\tau_{xz}u_x + \tau_{zz}u_z] = 0 \tag{A.2}$$

The energy density equation $E = ½\tau_{ij}u_{i,j} + ½\rho\dot{u}_i\dot{u}_i$ (Synge, 1956-1957) becomes

$$2E = \tau_{xx}u_{x,x} + \tau_{xz}u_{x,z} + \tau_{xz}u_{z,x} + \tau_{zz}u_{z,z} + ½\rho\dot{u}_x\dot{u}_x + ½\rho\dot{u}_z\dot{u}_z \tag{A.3}$$

Performing the x-derivatives yields:

$$2E = ik[\tau_{xx}u_x + \tau_{xz}u_z] + \left[\tau_{xz}u_{x,z} + \tau_{zz}u_{z,z}\right] - \rho\omega^2[u_x u_x + u_z u_z] \tag{A.4}$$

Substituting Eqn. (A.1) yields

$$2E = \frac{k}{\omega} e_x + \left[\tau_{xz} u_{x,z} + \tau_{zz} u_{z,z}\right] - \rho\omega^2 [u_x u_x + u_z u_z] \tag{A.5}$$

We now differentiate Eqn. (A.2) with respect to z:

$$\begin{gathered} [\tau_{xz} u_x + \tau_{zz} u_z]_{,z} = 0 = \left[\tau_{xz} u_{x,z} + \tau_{zz} u_{z,z}\right] + \left[\tau_{xz,z} u_x + \tau_{zz,z} u_z\right] \\ \left[\tau_{xz} u_{x,z} + \tau_{zz} u_{z,z}\right] = -\left[\tau_{xz,z} u_x + \tau_{zz,z} u_z\right] \end{gathered} \tag{A.6}$$

Substituting this result into Eqn. (A.5) yields

$$2E = \frac{k}{\omega} e_x - \left[\tau_{xz,z} u_x + \tau_{zz,z} u_z\right] - \rho\omega^2 [u_x u_x + u_z u_z] \tag{A.7}$$

The equations of motion are

$$\begin{gathered} -\rho\omega^2 u_x = \tau_{xx,x} + \tau_{xz,z} \\ -\rho\omega^2 u_z = \tau_{xz,x} + \tau_{zz,z} \end{gathered} \tag{A.8}$$

Summing them yields:

$$\begin{gathered} -\rho\omega^2 [u_x u_x + u_z u_z] = \left[\tau_{xz,z} u_x + \tau_{zz,z} u_z\right] + \left[\tau_{xx,x} u_x + \tau_{xz,x} u_z\right] \\ -\left[\tau_{xz,z} u_x + \tau_{zz,z} u_z\right] = \rho\omega^2 [u_x u_x + u_z u_z] + \left[\tau_{xx,x} u_x + \tau_{xz,x} u_z\right] \end{gathered} \tag{A.9}$$

Substituting this result into Eqn. (A.7) yields

$$2E = \frac{k}{\omega} e_x + \left[\tau_{xx,x} u_x + \tau_{xz,x} u_z\right] \tag{A.10}$$

Performing the x derivatives yield

$$2E = \frac{k}{\omega} e_x + ik[\tau_{xx} u_x + \tau_{xz} u_z] \tag{A.11}$$

Substituting Eqn. (A.1) yields

$$2E = \frac{k}{\omega} e_x + ik \frac{e_x}{i\omega} = 2\frac{k}{\omega} e_x$$

(A.12)

Introducing the horizontal velocity a $v_H = \omega/k$, we arrive at the result that the horizontal energy flux is the energy density multiplied by the horizontal velocity.

$$e_x = v_H E$$

(A.13)

Synge, J.L., 1956-1957, Flux of energy for elastic waves in anisotropic media, Proceedings of the Royal Irish Academy: Section A, Mathematical and Physical Sciences 58, 12-21.