Ray Theoretical Transport Equation for Two Seismically-Relevant Scalar Wave Equations

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Many textbooks derive ray theory for the wave equation $D\ddot{\varphi} = M\nabla^2\varphi$, where φ is the scalar field, D is density, M is elastic modulus and velocity is $c = M^{½}D^{-½}$. However, this equation is not a good analogue of the vector seismic wave equation, because it does not arise from the underlying physics when both D and M vary with position. For instance, for acoustic waves satisfy $\ddot{\varphi} = MD^{-1}\nabla^2\varphi + M\nabla D^{-1} \cdot \nabla\varphi$ where φ is pressure and $M = \lambda$ is incompressibility and, under appropriate assumptions, vertically-polarized shear waves satisfy $\ddot{\varphi} = MD^{-1}\nabla^2\varphi + D^{-1}\nabla M \cdot \nabla\varphi$, where φ is vertical displacement and and $M = \mu$ is rigidity.

Ray theory assumes that the wave field can be written as the Laurent series in angular frequency ω

$$\varphi(\mathbf{x}, t) = \sum_{n=0}^{\infty} (i\omega)^{-n} A_n(\mathbf{x}) \exp\{i\omega T(\mathbf{x}) - i\omega t\}$$

involving travel time function $T(\mathbf{x})$ and amplitudes $A_n(\mathbf{x})$ (with only A_0 important at high frequencies). Relevant derivatives of this function are

$$\ddot{\varphi}(\mathbf{x}, t) = \sum_{n=0}^{\infty} (i\omega)^{-n} (i\omega)^2 A_n(\mathbf{x}) \exp\{i\omega T(\mathbf{x}) - i\omega t\}$$

$$\nabla\varphi = \sum_{n=0}^{\infty} (i\omega)^{-n} \{\nabla A_n + (i\omega) A_n \nabla \mathrm{T}\} e^{i\omega T - i\omega t}$$

$$\nabla^2\varphi = \sum_{n=0}^{\infty} (i\omega)^{-n} \{\ \nabla^2 A_n + (i\omega)\nabla A_n \cdot \nabla \mathrm{T} + (i\omega)\nabla A_n \cdot \nabla \mathrm{T} + (i\omega) A_n \nabla^2 \mathrm{T} + (i\omega)^2 A_n \nabla \mathrm{T} \cdot \nabla \mathrm{T}\ \} e^{i\omega T - i\omega t}$$

Case 1: We insert the series into the wave equation

$$\ddot{\varphi} = c^2\nabla^2\varphi + M\nabla D^{-1} \cdot \nabla\varphi \quad \text{with} \quad c^2 = \frac{M}{D}$$

And equate powers of frequency. The $(i\omega)^2$ term is

$$A_0 = c^2 A_0 \nabla \mathrm{T} \cdot \nabla \mathrm{T}$$

which leads to the Eikonal equation $c^2\nabla \mathrm{T} \cdot \nabla \mathrm{T} = 1$. The $(i\omega)^1$ term is

$$A_1 = 2c^2\nabla A_0 \cdot \nabla \mathrm{T} + c^2 A_0 \nabla^2 \mathrm{T} + c^2 A_1 \nabla \mathrm{T} \cdot \nabla \mathrm{T} + A_0 \mathrm{M} \nabla D^{-1} \cdot \nabla \mathrm{T}$$

The Eikonal equation is used to remove term containing A_1, leaving the transport equation

$$0 = 2c^2\nabla A_0 \cdot \nabla \mathrm{T} + c^2 A_0 \nabla^2 \mathrm{T} + A_0 \mathrm{M} \nabla D^{-1} \cdot \nabla \mathrm{T}$$

This equation can be rearranged

$$0 = 2\nabla A_0 \cdot \nabla \mathrm{T} + A_0 \nabla^2 \mathrm{T} + c^{-2} A_0 \mathrm{M} \nabla D^{-1} \cdot \nabla \mathrm{T}$$

$$\left(\frac{2\nabla A_0}{A_0}\right) \cdot \nabla \mathrm{T} = -\nabla^2 \mathrm{T} - c^{-2} \mathrm{M} \nabla D^{-1} \cdot \nabla \mathrm{T}$$

Note that $\nabla A_0^2 = 2A_0 \nabla A_0$, so $2\nabla A_0 = \nabla A_0^2 / A_0$. We can also define a local ray direction $\mathbf{t}$ that is normal to a surface of equal travel time. From the Eikonal equation $\nabla \mathrm{T} = \mathbf{t}/c$. The transport equation then becomes:

$$\left(\frac{\nabla A_0^2}{A_0^2}\right) \cdot \left(\frac{\mathbf{t}}{c}\right) = -\nabla \cdot \left(\frac{\mathbf{t}}{c}\right) - c^{-2} \mathrm{M} \nabla D^{-1} \cdot \left(\frac{\mathbf{t}}{c}\right)$$

$$\left(\frac{\nabla A_0^2}{A_0^2}\right) \cdot \mathbf{t} = -c\nabla \cdot \left(\frac{\mathbf{t}}{c}\right) - c^{-1} \mathrm{M} \nabla D^{-1} \cdot \left(\frac{\mathbf{t}}{c}\right)$$

Applying the chain rule leads to

$$\left(\frac{\nabla A_0^2}{A_0^2}\right) \cdot \mathbf{t} = -\nabla \cdot \mathbf{t} - c\nabla \left(\frac{1}{c}\right) \cdot \mathbf{t} - c^{-2} \mathrm{M} \nabla D^{-1} \cdot \mathbf{t}$$

$$\left(\frac{\nabla A_0^2}{A_0^2}\right) \cdot \mathbf{t} = -\nabla \cdot \mathbf{t} + c^{-1} \nabla \mathrm{c} \cdot \mathbf{t} - \left(\frac{D}{M}\right) M \nabla D^{-1} \cdot \mathbf{t}$$

$$\left(\frac{\nabla A_0^2}{A_0^2}\right) \cdot \mathbf{t} = -\nabla \cdot \mathbf{t} + c^{-1} \nabla \mathrm{c} \cdot \mathbf{t} - D \nabla D^{-1} \cdot \mathbf{t}$$

Now let $A_0^2 = cDF$ where F will turn out to be energy flux.

$$\left(\frac{\nabla (cDF)}{cDF}\right) \cdot \mathbf{t} = -\nabla \cdot \mathbf{t} + c^{-1} \nabla \mathrm{c} \cdot \mathbf{t} - D \nabla D^{-1} \cdot \mathbf{t}$$

Applying the chain rule and the identity $B^{-1} \nabla \mathrm{B} = -B \nabla B^{-1}$

$$\left(\frac{cD\nabla F}{cDF}\right) \cdot \mathbf{t} + \left(\frac{FD\nabla c}{cDF}\right) \cdot \mathbf{t} + \left(\frac{cF\nabla D}{cDF}\right) \cdot \mathbf{t} = -\nabla \cdot \mathbf{t} + c^{-1} \nabla \mathrm{c} \cdot \mathbf{t} - D \nabla D^{-1} \cdot \mathbf{t}$$

$$\left(\frac{\nabla F}{F}\right) \cdot \mathbf{t} + \left(\frac{\nabla c}{c}\right) \cdot \mathbf{t} + \left(\frac{\nabla D}{D}\right) \cdot \mathbf{t} = -\nabla \cdot \mathbf{t} + c^{-1} \nabla \mathrm{c} \cdot \mathbf{t} + D^{-1} \nabla D \cdot \mathbf{t}$$

Canceling terms leads to

$$\left(\frac{\nabla F}{F}\right) \cdot \mathbf{t} = -\nabla \cdot \mathbf{t}$$

Defining $\mathbf{e} = F\mathbf{t}$, we note that this equation is equivalent to $\nabla \cdot \mathbf{e} = 0$, as

$$\nabla \cdot \mathbf{e} = 0 = \nabla \cdot (F\mathbf{t}) = \nabla F \cdot \mathbf{t} + F \nabla \cdot \mathbf{t}$$

So $F \propto c^{-1}D^{-1}A_0^2$ is a conserved flux, e.g. for acoustic waves, $F \propto c^{-1}\rho^{-1}P^2$. We have used the proportional sign because the $\nabla F/F$ term indicates that the flux is determined only up to an overall constant.

Case 2: We insert the series into the wave equation

$$\ddot{\varphi} = D^{-1}M\nabla^2\varphi + D^{-1}\nabla \mathrm{M} \cdot \nabla\varphi \quad \text{with} \quad c^2 = MD^{-1}$$

and equate equal powers of frequency. The $(i\omega)^2$ terms is

$$A_0 = MD^{-1}A_0\nabla \mathrm{T} \cdot \nabla \mathrm{T}$$

Which again leads to the Eikonal equation $c^{-2}\nabla \mathrm{T} \cdot \nabla \mathrm{T} = 1$. The $(i\omega)^1$ term is

$$A_1 = 2MD^{-1}\nabla A_0 \cdot \nabla \mathrm{T} + MD^{-1}A_0\nabla^2\mathrm{T} + MD^{-1}A_1\nabla \mathrm{T} \cdot \nabla \mathrm{T} + A_0D^{-1}\nabla \mathrm{M} \cdot \nabla \mathrm{T}$$

Subtract A_1 times Eikonal equation leads to a different transport equation than for Case 1

$$0 = 2MD^{-1}\nabla A_0 \cdot \nabla \mathrm{T} + MD^{-1}A_0\nabla^2\mathrm{T} + A_0D^{-1}\nabla \mathrm{M} \cdot \nabla \mathrm{T}$$

Multiply by $M^{-1}DA_0^{-1}$ yields

$$0 = \left(\frac{2\nabla A_0}{A_0}\right) \cdot \nabla \mathrm{T} + \nabla^2\mathrm{T} + M^{-1}\nabla \mathrm{M} \cdot \nabla \mathrm{T}$$

Inserting the identity $2\nabla A_0 = \nabla A_0^2/A_0$ yields

$$\left(\frac{\nabla A_0^2}{A_0^2}\right) \cdot \nabla \mathrm{T} = -\nabla \cdot \nabla \mathrm{T} - M^{-1}\nabla \mathrm{M} \cdot \nabla \mathrm{T}$$

We then introduce rat direction $\mathbf{t}$, which satisfies $\nabla \mathrm{T} = \mathbf{t}/c = D^{½}\mathbf{t}/M^{½}$

$$\left(\frac{\nabla A_0^2}{A_0}\right) \cdot \frac{D^{½}\mathbf{t}}{M^{½}} = -\nabla \cdot \left(\frac{D^{½}\mathbf{t}}{M^{½}}\right) - M^{-1}\nabla \mathrm{M} \cdot \left(\frac{D^{½}\mathbf{t}}{M^{½}}\right)$$

Multiplying by $M^{½}D^{-½}$ yields

$$\left(\frac{\nabla A_0^2}{A_0}\right) \cdot \mathbf{t} = -c\nabla \cdot (c\mathbf{t}) - M^{-½}D^{-½}\nabla \mathrm{M} \cdot (M^{-½}D^{½}\mathbf{t})$$

We apply the chain rule and simplify

$$\left(\frac{\nabla A_0^2}{A_0}\right) \cdot \mathbf{t} = -\nabla \cdot \mathbf{t} - c\nabla c^{-1} \cdot \mathbf{t} - M^{-1}\nabla \mathrm{M} \cdot \mathbf{t}$$

And apply the identities $M\nabla M^{-1} = -M^{-1}\nabla \mathrm{M}$ and $c^{-1}\nabla \mathrm{c} = -c\nabla c^{-1}$. Yielding

$$\left(\frac{\nabla A_0^2}{A_0}\right) \cdot \mathbf{t} = -\nabla \cdot \mathbf{t} + c^{-1}\nabla \mathrm{c} \cdot \mathbf{t} + M\nabla M^{-1} \cdot \mathbf{t}$$

We define $A_0^2 = cM^{-1}F$ and apply the chain rule

$$\left(\frac{\nabla(cM^{-1}F)}{cM^{-1}F}\right)\cdot\mathbf{t} = -\nabla\cdot\mathbf{t} + c^{-1}\nabla \mathrm{c}\cdot\mathbf{t} + M\nabla M^{-1}\cdot\mathbf{t}$$

$$\left(\frac{\nabla F}{F}\right)\cdot\mathbf{t} + c^{-1}\nabla c\cdot\mathbf{t} + M\nabla M^{-1}\cdot\mathbf{t} = -\nabla\cdot\mathbf{t} + c^{-1}\nabla \mathrm{c}\cdot\mathbf{t} + M\nabla M^{-1}\cdot\mathbf{t}$$

And cancel terms

$$\left(\frac{\nabla F}{F}\right)\cdot\mathbf{t} = -\nabla\cdot\mathbf{t}$$

Consequently, the conserved flux is $F = c^{-1}MA_0^2 = D^{½}M^{-½}MA_0^2 = D^{½}M^{½}A_0^2$.

For shear waves $M = \mu$ and the flux is $F \propto \rho^{½}\mu^{½}U_z^2$, where as in Case 1, the flux is determined only up to a multiplicative constant.