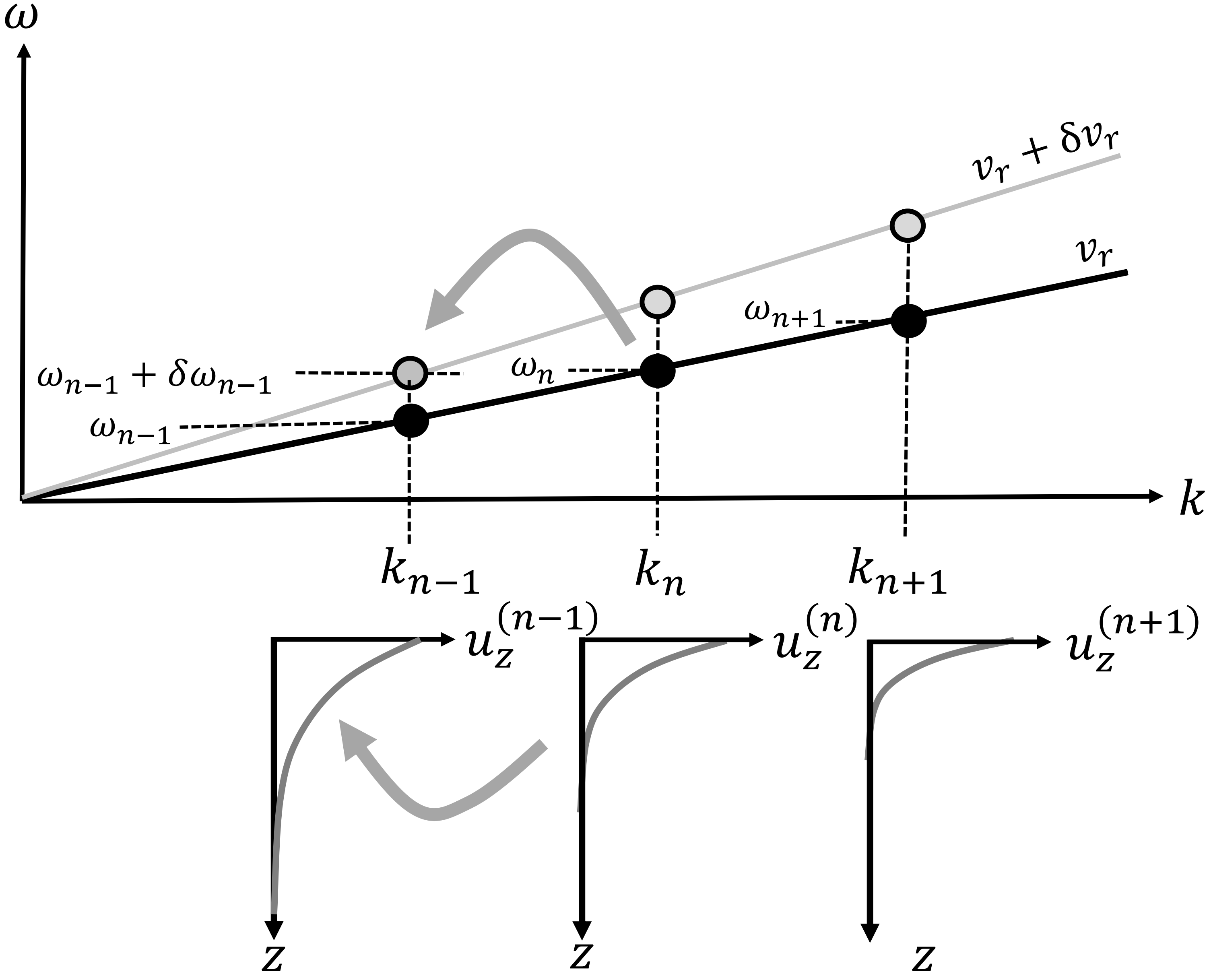
Perturbation Theory Analysis of Rayleigh Wave Modes, Bill Menke. June 10, 2024

(1) We are considering Rayleigh wave propagation in a uniform half-space.
(2) We quantize the horizontal wavenumber $k_n = \pm 2\pi n/L$ by making the model of finite length L with repeating boundary conditions. Because of the $\pm$, the modes occur as degenerate pairs, one right-propagating and the other left-propagating. However, nothing we will do here couples the two directions, so we can just ignore, say, the left-propagating modes.
(3) The horizontal wave-functions are $\exp(ik_n x)$. The vertical wave-functions $u_x^{(n)}(z)$ and $u_z^{(n)}(z)$ decreases with depth, with the lower wavenumbers extending deeper into the Earth model than the higher wavenumbers.
(4) Modal frequencies ω_n lie on line $\omega_n/k_n = v_r$ with Rayleigh phase velocity v_r.
(5) Spatially-uniform perturbations in α, β, ρ can be chosen that lead to an increase in phase velocity to $v_r + \delta v_r$. This perturbation does not change the wavenumber of any mode; it is always $k_n = 2\pi n/L$, because wavenumber is controlled by the geometry of the model.
(6) Frequencies of modes change to $\omega_n + \delta\omega_n$, which lie on a line with phase velocity $v_r + \delta v_r$
(7) No mode mixing occurs, because all the overlap integrals are zero for a spatially-uniform perturbation; the modes preserve their shapes even though their frequencies change.
(8) For some sufficiently large perturbation, $\omega_{n-1} + \delta\omega_{n-1} = \omega_n$. This can be interpreted as the wavenumber of the mode with frequency ω_n "decreasing" from k_n to k_{n-1}.
(9) And the corresponding vertical wave-function extends deeper into the Earth model.
(10) As the length of the model L is increased, the mode spacing of the modes decreases, so the effect of perturbations of any size can be assessed by choosing a model of appropriate length L.



ω
$v_r + \delta v_r$
v_r
ω_{n+1}
$\omega_{n-1} + \delta\omega_{n-1}$
ω_n
ω_{n-1}
k
k_{n-1}
k_n
k_{n+1}
$u_z^{(n-1)}$
$u_z^{(n)}$
$u_z^{(n+1)}$
z
z
z

Continuum limit

$$\omega_n = \omega_{n-1} + \delta\omega_{n-1}$$

$$\omega_n - \omega_{n-1} = \delta\omega_{n-1}$$

approximate $\omega_n = \omega_{n-1} - \left.\frac{d\omega}{dk}\right|_{n-1} \Delta k$

define Δk as the shift from n to n-1 hence the minus sign

$$(\omega_{n-1} - \left.\frac{d\omega}{dk}\right|_{n-1} \Delta k - \omega_{n-1} = \delta\omega_{n-1}$$

$$-\left.\frac{d\omega}{dk}\right|_{n-1} \Delta k = \delta\omega_{n-1}$$

$$\Delta k = -\left[\left.\frac{d\omega}{dk}\right|_{n-1}\right]^{-1} \delta\omega_{n-1}$$

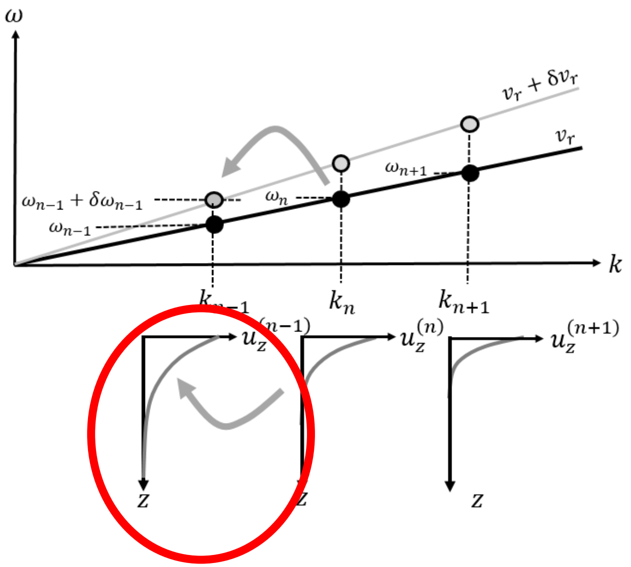
Note that formula involves group velocity $d\omega/dk$

We rewrite this relationship as

$$\Delta k = -\left[\left.\frac{d\omega}{dk}\right|_{k_0}\right]^{-1} \frac{\partial[\delta\omega_{k_0}]}{\partial m_1} \Delta m_1$$

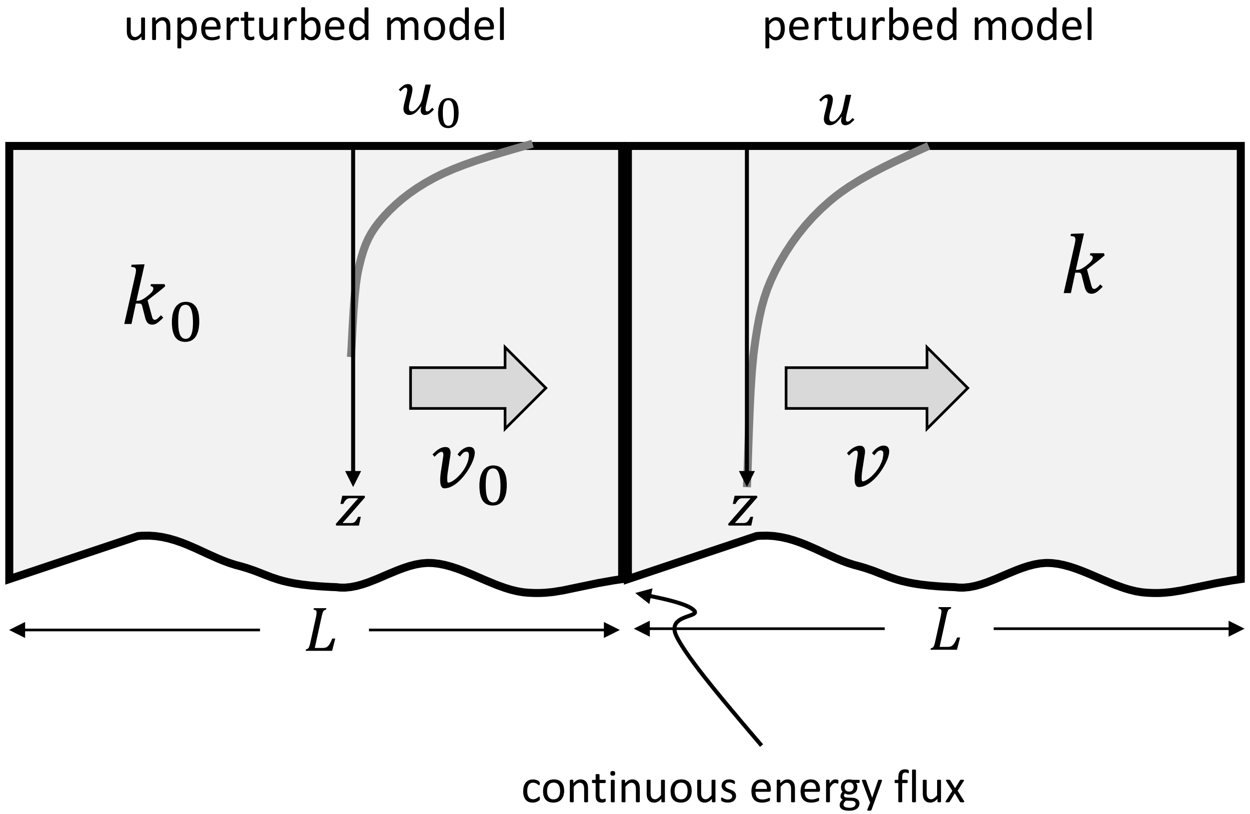
where m_1 is a material property; that is, a change in material property shifts modal frequencies leading to a change in wavenumber Δk from the reference value k_0 at constant frequency.

Vertically-stratified perturbation when mode mixing occurs. The analysis in the diagram basically is the same, except that the circled part of the plot:



needs to be interpreted as $u_z^{(n-1)} + \delta u_z^{(n-1)}$; that is the perturbed wave-function. The mode mixture will contain not only other modes on the fundamental branch, but higher modes, too (though their contribution will be small).

constant frequency ω



My idea is to consider separate unperturbed and perturbed models positioned side by side. If the amplitude of the perturbed mode is chosen properly, then the energy flux at the join is continuous, and energy "appear" to flow from the left model into the right model. Thus, the two models act as one single model.

1. Frequency ω is being held constant.
2. The amplitude of the **modes** are A_0 and A
3. The total energy of a mode of unit amplitude is chosen to be $E_{T0} = E_T = \omega^2$
4. As energy of a mode is evenly distributed horizontally, the vertically-integrated energy density of a mode is ω^2/L
5. The vertically integrated horizontal energy flux is then ω^2/L times the velocity in the box.
6. The fluxes match when $\omega^2 v_0 A_0^2 = \omega^2 v A^2$.
7. Note that density does not explicitly appear in this equation, as it does in the shear wave case, $\omega^2 \rho_0 v_0 A_0^2 = \omega^2 \rho v A^2$, but that's because it appears in the energy normalization of the mode. In essence, density is "still there". (See note on last page).

1. For the Rayleigh wave mode, a perturbation in material properties leads to a change in wavenumber Δk.
2. At constant frequency, velocity shifts to $v = \omega/k = \omega(k_0 + \Delta k)^{-1} = v_0(1 + \Delta k/k_0)^{-1} = v_0(1 - \Delta k/k_0)$
3. So

$$\omega^2 v_0 A_0^2 = \omega^2 v A^2$$

$$v_0 A_0^2 = v_0(1 - \Delta k/k_0)A^2$$

$$A_0^2 = (1 - \Delta k/k_0)A^2$$

$$A^2 = (1 + \Delta k/k_0)A_0^2$$

$$\frac{(A_0 + \Delta A)^2}{A_0^2} = (1 + \Delta k/k_0)$$

$$\left(1 + \frac{\Delta A}{A_0}\right)^2 = (1 + \Delta k/k_0)$$

$$\left(1 + \frac{2\Delta A}{A_0}\right) = (1 + \Delta k/k_0)$$

$$\frac{\Delta A}{A_0} = ½\,\frac{\Delta k}{k_0}$$

with $\Delta k = -\left[\left.\frac{d\omega}{dk}\right|_{k_0}\right]^{-1} \frac{\partial[\delta\omega_{k_0}]}{\partial m_1}\Delta m_1$

So, the change in amplitude involves the group velocity.

4. However, keep in mind that this is the change in amplitude of the **mode** and not the change in amplitude of the **vertical component of displacement at $z = 0$**

To get the latter you need to add two more derivatives

$$\Delta U_z = \left.\frac{\partial U_z}{\partial A}\right|_k \Delta A + \left.\frac{\partial U_z}{\partial k}\right|_A \Delta k$$

and we have not yet devised an (easy) way t calculate it.

Note: Modes of the seismic wave equation obey a type of orthogonality in which density ρ enters as a weighting factor in the volume integral:

$$\iiint \rho u_i^{(n)} u_i^{(m)*} dV = \delta_{nm} \quad \text{and} \quad \iiint \tau_{ij}^{(n)} u_{i,j}^{(n)*} dV = \omega_n^2 \delta_{nm}$$

The total energy in a properly normalized mode is

$$E_T \equiv \iiint \left(½\tau_{pq}^{A} u_{p,q}^{A*} + ½\omega_A^2 \rho u_p^A u_p^{A*} \right) dV = ½\omega_A^2 + ½\omega_A^2 = \omega_A^2$$