Properties of modes of the possibly-anisotropic seismic wave equation

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1. Wave equation for modes (presuming homogeneous boundary conditions)

$$-\omega_A^2 \rho u_i^A = \tau_{ij,j}^A = c_{ijrs} u_{rs,j}^A$$

$$-\omega_B^2 \rho u_p^{B*} = \tau_{pq,q}^{B*} = c_{pqrs} u_{rs,q}^{B*}$$

2. Identity A ("integration by parts")

$$\left(u_i^A \tau_{pq}^{B*}\right)_{,q} = u_{i,q}^A \tau_{pq}^{B*} + u_i^A \tau_{pq,q}^{B*}$$

$$\iiint \left(u_i^A \tau_{pq}^{B*}\right)_{,q} dV = \iiint u_{i,q}^A \tau_{pq}^{B*} dV + \iiint u_i^A \tau_{pq,q}^{B*} dV$$

$$\iint u_i^A \tau_{pq}^{B*} n_q dS = \iiint u_{i,q}^A \tau_{pq}^{B*} dV + \iiint u_i^A \tau_{pq,q}^{B*} dV$$

$\iint u_i^A \tau_{pq}^{B*} n_q dS = \iint u_i^A T_p^{B*}(\mathbf{n}) dS = 0$ (homogeneous BC's)

$$\iiint u_i^A \tau_{pq,q}^{B*} dV = -\iiint u_{i,q}^A \tau_{pq}^{B*} dV$$

3. Identity B

Consider

$\iiint u_{p,q}^A \tau_{pq}^{B*} dV$ and $\iiint u_{p,q}^{B*} \tau_{pq}^A dV$

insert $\tau_{pq}^A = c_{pqrs} u_{r,s}^A$ and $\tau_{pq}^{B*} = c_{pqrs} u_{r,s}^{B*}$

$\iiint c_{pqrs} u_{p,q}^A u_{r,s}^{B*} dV$ and $\iiint c_{pqrs} u_{p,q}^{B*} u_{r,s}^A dV$

apply symmetry $c_{pqrs} = c_{rspq}$ and swap names (pq) and (rs)

$\iiint c_{pqrs} u_{p,q}^A u_{r,s}^{B*} dV$ and $\iiint c_{pqrs} u_{p,q}^A u_{r,s}^{B*} dV$

and note that these are equal, so

$$\iiint u_{p,q}^A \tau_{pq}^{B*} dV = \iiint u_{p,q}^{B*} \tau_{pq}^A dV$$

4. Identity C ("Orthogonality of modes, part A")

Multiply wave equations $\omega_A^2 \rho u_i^A = \tau_{ij,j}^A$ and $\tau_{pq,q}^{B*} = \omega_B^2 \rho u_p^{B*}$

$$\omega_A^2 \rho u_i^A \tau_{pq,q}^{B*} = \omega_B^2 \rho u_p^{B*} \tau_{ij,j}^A$$

Divide through by ρ

$$\omega_A^2 u_i^A \tau_{pq,q}^{B*} = \omega_B^2 u_p^{B*} \tau_{ij,j}^A$$

$$\omega_A^2 u_i^A \tau_{pq,q}^{B*} - \omega_B^2 u_p^{B*} \tau_{ij,j}^A = 0$$

$$\omega_A^2 \iiint u_i^A \tau_{pq,q}^{B*} dV - \omega_B^2 \iiint u_p^{B*} \tau_{iq,q}^A dV = 0$$

Apply identity A

$$-\omega_A^2 \iiint u_{i,q}^A \tau_{pq}^{B*} dV + \omega_B^2 \iiint u_{p,q}^{B*} \tau_{iq}^A dV = 0$$

Apply Identity B

$$-\omega_A^2 \iiint u_{i,q}^A \tau_{pq}^{B*} dV + \omega_B^2 \iiint u_{i,q}^A \tau_{pq}^{B*} dV = 0$$

So

$$(\omega_A^2 - \omega_B^2) \iiint u_{p,q}^A \tau_{pq}^{B*} dV = 0$$

5. Identity D ("Orthogonality of modes, part B")

$-\omega_A^2 \rho u_i^A = \tau_{ij,j}^A$ and $-\omega_B^2 \rho u_p^{B*} = \tau_{pq,q}^{B*}$

multiply by first by u_p^B and second by u_i^A, contract (i, p) and rename j to q

$-\omega_A^2 \rho u_p^A u_p^{B*} = u_p^{B*} \tau_{pq,q}^A$ and $-\omega_B^2 \rho u_p^A u_p^{B*} = u_p^A \tau_{pq,q}^{B*}$

subtract and integrate

$$\omega_A^2 \rho u_p^A u_p^{B*} - \omega_B^2 \rho u_p^A u_p^{B*} = -u_p^{B*} \tau_{pq,q}^A + u_p^A \tau_{pq,q}^{B*}$$

integrate

$$(\omega_A^2 - \omega_B^2) \iiint \rho u_p^A u_p^{B*} dV = - \iiint u_p^{B*} \tau_{pq,q}^A dV + \iiint u_p^A \tau_{pq,q}^{B*} dV$$

apply identity A

$$(\omega_A^2 - \omega_B^2) \iiint \rho u_p^A u_p^{B*} dV = \iiint u_{p,q}^{B*} \tau_{pq}^A dV - \iiint u_{p,q}^A \tau_{pq}^{B*} dV$$

apply identity B

$$(\omega_A^2 - \omega_B^2) \iiint \rho u_p^A u_p^{B*} dV = \iiint u_{p,q}^{B*} \tau_{pq}^A dV - \iiint u_{p,q}^{B*} \tau_{pq}^A dV = 0$$

$$(\omega_A^2 - \omega_B^2) \iiint \rho u_p^A u_p^{B*} dV = 0$$

6. Normalization of modes

Start with the wave equation for mode n

$-\omega_n^2 \rho u_i^{(n)} = \tau_{ij,j}^{(n)}$ (with homogeneous boundary conditions)

Multiply by $u_i^{(n)*}$ and integrate

$$-\omega_n^2 \iiint \rho u_i^{(n)} u_i^{(n)*} dV = \iiint \tau_{ij,j}^{(n)} u_i^{(n)*} dV$$

Apply Identity A

$$-\omega_n^2 \iiint \rho u_i^{(n)} u_i^{(n)*} dV = -\iiint \tau_{ij}^{(n)} u_{i,j}^{(n)*} dV$$

Choose amplitude of $u_i^{(n)}$ so that

$$\iiint \rho u_i^{(n)} u_i^{(n)*} dV = 1$$

Then

$$\iiint \tau_{ij}^{(n)} u_{i,j}^{(n)*} dV = \omega_n^2$$

Proof that results are real relies on fact that $f = f^*$ only when f is real, together with $(fg)^* = f^* g^*$ and $f = f^{**}$

$$\left[\iiint \rho u_i^{(n)} u_i^{(n)*} dV\right]^* = \iiint \rho u_i^{(n)*} u_i^{(n)} dV = \iiint \rho u_i^{(n)} u_i^{(n)*} dV$$

$$\left[\iiint \tau_{ij}^{(n)} u_{i,j}^{(n)*} dV\right]^* = \left[\iiint c_{ijpq} u_{p,q}^{(n)} u_{i,j}^{(n)*} dV\right]^* = \iiint c_{ijpq} u_{p,q}^{(n)*} u_{i,j}^{(n)} dV$$

$$= \iiint c_{pqij} u_{i,j}^{(n)} u_{p,q}^{(n)*} dV = \iiint c_{ijpq} u_{p,q}^{(n)} u_{i,j}^{(n)*} dV = \iiint \tau_{ij}^{(n)} u_{i,j}^{(n)*} dV$$

Here we have used $c_{pqij} = c_{ijpq}$ and interchanged the names of the dummy variables (i,j) and (p,q).

7. Lagrangian

From the formula from above

$$2L \equiv \omega_n^2 \iiint \rho u_i^{(n)} u_i^{(n)*} dV - \iiint \tau_{ij}^{(n)} u_{i,j}^{(n)*} dV = 0$$

The term involving $\omega_n^2 \rho u_i^{(n)} u_i^{(n)*}$ is twice the kinetic energy of the mode and the term involving $\tau_{ij}^{(n)} u_{i,j}^{(n)*}$ is twice its potential energy. The equation indicate that they are equal. The difference between the kinetic and potential energy is the Lagrangian, L.

The Lagrangian is stationary with respect to a perturbation $\delta u_i^{(n)}$ of the mode (as long as the perturbation obeys homogeneous boundary conditions).

$$2L + 2\delta L = \omega_n^2 \iiint \rho\left(u_i^{(n)} + \delta u_i^{(n)}\right)\left(u_i^{(n)*} + \delta u_i^{(n)*}\right) dV - \iiint \left(\tau_{ij}^{(n)} + \delta\tau_{ij}^{(n)}\right)\left(u_{i,j}^{(n)*} + \delta u_{i,j}^{(n)*}\right) dV =$$

$$2\delta L = \omega_n^2 \iiint \rho\left(u_i^{(n)} \delta u_i^{(n)*} + u_i^{(n)*} \delta u_i^{(n)}\right) dV - \iiint \left(\tau_{ij}^{(n)} \delta u_{i,j}^{(n)*} + \delta\tau_{ij}^{(n)} u_{i,j}^{(n)*}\right) dV$$

Using the complex conjugate of Identity A with $p = i, q = j, \tau_{ij}^B = \tau_{ij}^{(n)}, u_i^A = \delta u_{i,j}^{(n)}$

$\iiint \tau_{pq,q}^B u_i^{A*} dV = -\iiint \tau_{pq}^B u_{i,q}^{A*} dV$ becomes $\iiint \tau_{ij,j}^{(n)} \delta u_i^{(n)*} dV = -\iiint \tau_{ij}^{(n)} \delta u_{i,j}^{(n)*} dV$

Using Identity B with $p = i, q = j, \tau_{ij}^A = \delta\tau_{ij}^{(n)}, u_{i,j}^B = u_{i,j}^{(n)}$

$\iiint \tau_{ij}^{B*} u_{i,j}^A dV = \iiint \tau_{ij}^A u_{i,j}^{B*} dV$ becomes $\iiint \tau_{ij}^{(n)*} \delta u_{i,j}^{(n)} dV = \iiint \delta\tau_{ij}^{(n)} u_{i,j}^{(n)*} dV$

Applying Identity A yields

$$-\iiint \tau_{ij,j}^{(n)*} \delta u_i^{(n)} dV = \iiint \tau_{ij}^{(n)*} \delta u_{i,j}^{(n)} dV = \iiint \delta\tau_{ij}^{(n)} u_{i,j}^{(n)*} dV$$

Hence,

$$2\delta L = \omega_n^2 \iiint \rho \left(u_i^{(n)} \delta u_i^{(n)*} + u_i^{(n)*} \delta u_i^{(n)} \right) dV + \iiint \left(\tau_{ij,j}^{(n)} \delta u_i^{(n)*} + \tau_{ij,j}^{(n)*} \delta u_i^{(n)} \right) dV =$$

$$2\delta L = \iiint \left\{ \left(\omega_n^2 \rho u_i^{(n)} + \tau_{ij,j}^{(n)} \right) \delta u_i^{(n)*} + \left(\omega_n^2 \rho u_i^{(n)*} + \tau_{ij,j}^{(n)*} \right) \delta u_i^{(n)} \right\} dV = 0$$

As the equation of motion indicates that $\omega_n^2 \rho u_i^{(n)} + \tau_{ij,j}^{(n)} = 0$ (and likewise its complex conjugate).

8. Orthonormality

$$\iiint \rho u_i^{(n)} u_i^{(m)*} dV = \delta_{nm} \quad \text{and} \quad \iiint \tau_{ij}^{(n)} u_{i,j}^{(n)*} dV = \omega_n^2 \delta_{nm}$$

9. Total Energy

$$E_T \equiv \iiint E dV \quad \text{with} \quad E \equiv ½\tau_{pq} u_{p,q}^* + ½\rho \dot{u}_p \dot{u}_p^* = ½\tau_{pq} u_{p,q}^* + ½\omega^2 \rho u_p u_p^*$$

The total energy in a properly normalized mode is

$$E_T \equiv \iiint \left(½\tau_{pq}^A u_{p,q}^{A*} + ½\omega_A^2 \rho u_p^A u_p^{A*} \right) dV = ½\omega_A^2 + ½\omega_A^2 = \omega_A^2$$

Suppose two modes that $u_p = A u_p^A + B u_p^B$ and $\tau_{pq} = A\tau_{pq}^A + B\tau_{pq}^B$. (That u_p^A and τ_{pq}^A have the same scaling follows from the wave equation; that is, if $-\omega_A^2 \rho u_i^A = \tau_{ij,j}^B$ then $-\omega_A^2 \rho \left(A u_i^A \right) = \left(A\tau_{ij}^A \right)_{,j}$. So the energy density is

$$2E = \left[A\tau_{pq}^A + B\tau_{pq}^B \right]\left[A^* u_{p,q}^{A*} + B^* u_{p,q}^{B*} \right] + \rho\left[A\dot{u}_p^A + B\dot{u}_p^B \right]\left[A^* \dot{u}_p^A + B^* \dot{u}_p^B \right]$$

$$= AA^* \tau_{pq}^A u_{p,q}^{A*} + B^* B \tau_{pq}^B u_{p,q}^{B*} + AB^* \tau_{pq}^A u_{p,q}^{B*} + A^* B \tau_{pq}^B u_{p,q}^{A*}$$

$$+ \omega_A^2 \rho AA^* u_p^A u_p^{A*} + \omega_B^2 \rho B^* B u_p^B u_p^{B*} + \rho \omega_A \omega_B AB^* u_p^A u_p^{B*} + \rho \omega_A \omega_B A^* B u_p^{A*} u_p^B$$

Upon taking the volume integral, the 3rd, 4th, 7th and 8th terms become zero, leaving

$E_T = E_T^A + E_T^B$ with

$$E_T^A = AA^* \iiint \left[½\tau_{pq}^A u_{p,q}^{A*} + ½\rho\omega_A^2 u_p^A u_p^{A*} \right] dV = \omega_A^2 AA^*$$

$$E_T^A = BB^* \iiint \left[½\tau_{pq}^A u_{p,q}^{A*} + ½\rho\omega_A^2 u_p^A u_p^{A*} \right] dV = \omega_B^2 BB^*$$

That is, the total energy is the sum of the energy in the modes.

10. Non-degenerate perturbation theory

Unperturbed equation

$$-\omega_n^2 \rho u_i^{(n)} = \tau_{ij,j}^{(n)} = c_{ijpq} u_{p,qj}^{(n)}$$

Perturbed equation

$$-(\omega_n + \delta\omega_n)^2 (\rho + \delta\rho) \left(u_i^{(n)} + \delta u_i^{(n)} \right) \\ = \left(c_{ijpq} + \delta c_{ijrs} \right) \left(u_{p,qj}^{(n)} + \delta u_{p,qj}^{(n)} \right) + \left(c_{ijpq,j} + \delta c_{ijrs,j} \right) \left(u_{p,q}^{(n)} + \delta u_{p,q}^{(n)} \right)$$

Keep only zeroth and first order terms, noting $(\omega_n + \delta\omega_n)^2 \approx (\omega_n^2 + 2\omega_n \delta\omega_n)$

$$-(\omega_n^2 + 2\omega_n \delta\omega_n)(\rho + \delta\rho) \left(u_i^{(n)} + \delta u_i^{(n)} \right)$$

$$= \left(c_{ijpq} + \delta c_{ijrs} \right) \left(u_{p,qj}^{(n)} + \delta u_{p,qj}^{(n)} \right) + \left(c_{ijpq,j} + \delta c_{ijrs,j} \right) \left(u_{p,q}^{(n)} + \delta u_{p,q}^{(n)} \right)$$

$$-\omega_n^2 \rho u_i^{(n)} - \omega_n^2 \rho \delta u_i^{(n)} - (2\rho\omega_n \delta\omega_n + \omega_n^2 \delta\rho) u_i^{(n)}$$

$$= c_{ijpq} u_{p,qj}^{(n)} + c_{ijpq,j} u_{p,q}^{(n)} + c_{ijpq} \delta u_{p,qj}^{(n)} + \delta c_{ijpq} u_{p,qj}^{(n)} + \delta c_{ijrs,j} u_{p,q}^{(n)} + c_{ijpq,j} \delta u_{p,q}^{(n)}$$

Subtract zeroth order equation

$$-\omega_n^2 \rho \delta u_i^{(n)} - (2\rho\omega_n \delta\omega_n + \omega_n^2 \delta\rho) u_i^{(n)} = c_{ijpq} \delta u_{p,qj}^{(n)} + \delta c_{ijpq} u_{p,qj}^{(n)} + \delta c_{ijrs,j} u_{p,q}^{(n)} + c_{ijpq,j} \delta u_{p,q}^{(n)}$$

Rearrange

$$-\omega_n^2 \rho \delta u_i^{(n)} - c_{ijpq} \delta u_{p,qj}^{(n)} - c_{ijpq,j} \delta u_{p,q}^{(n)} = (2\rho\omega_n \delta\omega_n + \omega_n^2 \delta\rho) u_i^{(n)} + \delta c_{ijpq} u_{p,qj}^{(n)} + \delta c_{ijrs,j} u_{p,q}^{(n)}$$

Define $\delta\tau_{ij,j}^{(n)} = c_{ijpq} \delta u_{p,qj}^{(n)} + c_{ijpq,j} \delta u_{p,q}^{(n)}$ and rearrange

$$-\omega_n^2 \rho \delta u_i^{(n)} - \delta\tau_{ij,j}^{(n)} = (2\rho\omega_n \delta\omega_n + \omega_n^2 \delta\rho) u_i^{(n)} + \delta c_{ijpq} u_{p,qj}^{(n)} + \delta c_{ijrs,j} u_{p,q}^{(n)}$$

Define

$$\delta u_i^{(n)} \equiv \sum_{m \neq n} A_m \, u_i^{(m)}$$

$$\delta\tau_{ij,j}^{(n)} \equiv c_{ijpq} \delta u_{p,qj}^{(n)} + c_{ijpq,j} \delta u_{p,q}^{(n)} = \sum_{m \neq n} A_m \left(c_{ijpq} u_{p,qj}^{(m)} + c_{ijpq,j} u_{p,q}^{(m)} \right) = \sum_{m \neq n} A_m \, \tau_{ij,j}^{(m)}$$

So that

$$-\sum_{m\neq n} A_m\, \omega_n^2 \rho u_i^{(m)} - \sum_{m\neq n} A_m\, \delta\tau_{ij,j}^{(m)} = (\omega_n^2 \delta\rho + 2\rho\omega_n \delta\omega_n) u_i^{(n)} + \delta c_{ijpq} u_{p,qj}^{(n)} + \delta c_{ijpq,j} u_{p,q}^{(n)}$$

$$\sum_{m\neq n} A_m \left(\omega_n^2 \rho u_i^{(m)} + \tau_{ij,j}^{(m)}\right) = -(\omega_n^2 \delta\rho + 2\rho\omega_n \delta\omega_n) u_i^{(n)} - \delta c_{ijpq} u_{p,qj}^{(n)} - \delta c_{ijpq,j} u_{p,q}^{(n)}$$

Multiply by $u_i^{(k)*}$, sum over i, and perform volume integral $\langle f \rangle \equiv \iiint f dV$

$$\sum_{m\neq n} A_m \left(\omega_n^2 \langle \rho u_i^{(m)} u_i^{(k)*} \rangle + \langle \tau_{ij,j}^{(m)} u_i^{(k)*} \rangle\right) = -\omega_n^2 \langle \delta\rho u_i^{(n)} u_i^{(k)*} \rangle - 2\omega_n \delta\omega_n \langle \rho u_i^{(n)} u_i^{(k)*} \rangle$$

$$- \langle \delta c_{ijpq} u_{p,qj}^{(n)} u_i^{(k)*} \rangle - \langle \delta c_{ijpq,j} u_{p,q}^{(n)} u_i^{(k)*} \rangle$$

Apply Identity A

$$\sum_{m\neq n} A_m \left(\omega_n^2 \langle \rho u_i^{(m)} u_i^{(k)*} \rangle - \langle \tau_{ij}^{(m)} u_{i,j}^{(k)*} \rangle\right) = -\omega_n^2 \langle \delta\rho u_i^{(n)} u_i^{(k)*} \rangle - 2\omega_n \delta\omega_n \langle \rho u_i^{(n)} u_i^{(k)*} \rangle$$

$$- \langle \delta c_{ijpq} u_{p,qj}^{(n)} u_i^{(k)*} \rangle - \langle \delta c_{ijpq,j} u_{p,q}^{(n)} u_i^{(k)*} \rangle$$

Apply orthonormality

$$\sum_{m\neq n} A_m \left(\omega_n^2 \delta_{km} - \omega_k^2 \delta_{km}\right) = -\omega_n^2 \langle \delta\rho u_i^{(n)} u_i^{(k)*} \rangle - 2\omega_n \delta\omega_n \delta_{kn}$$

$$- \langle \delta c_{ijpq} u_{p,qj}^{(n)} u_i^{(k)*} \rangle - \langle \delta c_{ijpq,j} u_{p,q}^{(n)} u_i^{(k)*} \rangle$$

Case $k = n$ yields formula for $\delta\omega_n$

$$0 = -\omega_n^2 \langle \delta\rho u_i^{(n)} u_i^{(k)*} \rangle - 2\omega_n \delta\omega_n - \langle \delta c_{ijpq} u_{p,qj}^{(n)} u_i^{(k)*} \rangle - \langle \delta c_{ijpq,j} u_{p,q}^{(n)} u_i^{(k)*} \rangle$$

$$2\omega_n \delta\omega_n = -\omega_n^2 \langle \delta\rho u_i^{(n)} u_i^{(n)} \rangle - \langle \delta c_{ijpq} u_{p,qj}^{(n)} u_i^{(n)} \rangle - \langle \delta c_{ijpq,j} u_{p,q}^{(n)} u_i^{(n)} \rangle$$

Case $k \neq n$ yields formula for A_k

$$(\omega_n^2 - \omega_k^2) A_k = -\omega_n^2 \langle \delta\rho u_i^{(n)} u_i^{(k)*} \rangle - \langle \delta c_{ijpq} u_{p,qj}^{(n)} u_i^{(k)*} \rangle - \langle \delta c_{ijpq,j} u_{p,q}^{(n)} u_i^{(k)*} \rangle$$

Suppose a P-wave propagating in the x-direction in an isotropic medium with spatially constant perturbations

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \sin(ik_n x - i\omega_n t)$$

Then

$$\langle \delta\rho u_i^{(n)} u_i^{(n)} \rangle = \langle \delta\rho u_1^{(n)} u_1^{(n)} \rangle$$

$$\langle \delta c_{ijpq} u_{p,qj}^{(n)} u_i^{(n)} \rangle = \langle \delta c_{1111} u_{1,11}^{(n)} u_1^{(n)} \rangle = -k_n^2 \langle \delta c_{1111} u_1^{(n)} u_1^{(n)} \rangle$$

$$\delta c_{ijpq} = \delta\lambda \delta_{ij} \delta_{pq} + \delta\mu \delta_{ip} \delta_{jq} + \delta\mu \delta_{iq} \delta_{jp} \quad \text{so} \quad \delta c_{1111} = \delta\lambda + 2\delta\mu$$

$\langle \delta c_{ijpq,j} u_{p,q}^{(n)} u_i^{(k)*} \rangle = 0$

so

$2\omega_n \delta\omega_n = -\omega_n^2 \langle \delta\rho u_1^{(n)} u_1^{(n)*} \rangle + k_n^2 \langle (\delta\lambda + 2\delta\mu) u_1^{(n)} u_1^{(n)*} \rangle$

A density increases leads to a frequency decrease; an elastic moduli increase leads to a frequency increase, both of which make sense.

I am not so sure how to analyze the term $\langle \delta c_{ijpq,j} u_{p,q}^{(n)} u_i^{(k)*} \rangle$. At the very least one would like to show that

$$\langle \delta c_{ijpq} u_{p,qj}^{(n)} u_i^{(k)*} \rangle + \langle \delta c_{ijpq,j} u_{p,q}^{(n)} u_i^{(k)*} \rangle$$

is real. Identity A can be used to show that

$$\langle \delta c_{ijpq} u_{p,qj}^{(n)} u_i^{(k)*} \rangle + \langle \delta c_{ijpq,j} u_{p,q}^{(n)} u_i^{(k)*} \rangle = \langle \left(\delta c_{ijpq} u_{p,q}^{(n)} \right)_{,j} u_i^{(k)*} \rangle = \langle \delta c_{ijpq} u_{p,q}^{(n)} u_{i,j}^{(k)*} \rangle$$

at least in the cases where $\delta c_{ijpq} = 0$ on the boundary or $\delta c_{ijpq} \propto c_{ijpq}$ (where the surface integral vanishes). Then, following the same logic as in Sec. 5, the result is real a long as δc_{ijpq} is a proper perturbation to the elastic tensor in the sense of having the correct symmetries.