Appendix for

Frequency Dependence of Rayleigh Wave Amplification by Variation in Earth Structure Investigated Using the Constant Energy Flux Approximation

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A.1 Energy Density and Energy Flux Density. In a possibly-anisotropic elastic medium, particle displacement **u** satisfies the wave equation $\rho \ddot{u}_i = \tau_{ij,j}$ where $\tau_{ij} = c_{ijpq} u_{p,q}$ is the stress tensor (which is symmetric) and c_{ijpq} is the Hooke's law tensor (which has symmetries $c_{ijpq} = c_{jipq}$, $c_{ijpq} = c_{ijqp}$ and $c_{ijpq} = c_{pqij}$). Here, we have used Einstein notation in which repeated indices are summed, a comma indicates spatial differentiation $\partial/\partial x_i$ and a dot temporal differentiation $\partial/\partial t$. Synge (1956-1957) showed that energy density, E and an energy flux density **e** are

$$E = ½\tau_{ij} u_{i,j} + ½\rho \dot{u}_i \dot{u}_i \quad \text{and} \quad e_i = -\tau_{ij} \dot{u}_j \tag{A.1a,b}$$

such that the two are related by the standard conservation equation

$$\dot{E} = -e_{i,i} \tag{A.2}$$

This relationship can be proved by using the chain rule to take the divergence of the energy flux density

$$-e_{i,i} = \left(\tau_{ij} \dot{u}_j\right)_{,i} = \tau_{ij,i} \dot{u}_j + \tau_{ij} \dot{u}_{j,i} \tag{A.3}$$

where the last term arises from substituting the equation of motion. Thus,

$$-e_{i,i} = \rho \ddot{u}_i \dot{u}_i + \tau_{ij} \dot{u}_{j,i} = \frac{\partial}{\partial t}(½\rho \dot{u}_i \dot{u}_i) + \tau_{ij} \dot{u}_{j,i} \tag{A.4}$$

Inserting Hooke's law $\tau_{ij} = c_{ijpq} u_{p,q}$ yields $\tau_{ij} \dot{u}_{j,i} = c_{ijpq} u_{p,q} \dot{u}_{j,i}$. As long as c_{ijpq} is not a function of time, and utilizing the symmetry $c_{ijpq} = c_{pq,ij}$, we find

$$\frac{\partial}{\partial t}\left(\tau_{ij} u_{j,i}\right) = \frac{\partial}{\partial t}\left(½c_{ijpq} u_{p,q} u_{j,i}\right) = ½c_{ijpq} u_{p,q} \dot{u}_{j,i} + ½c_{ijpq} \dot{u}_{p,q} u_{j,i} =$$

$$½c_{ijpq} u_{p,q} \dot{u}_{j,i} + ½c_{ijpq} u_{p,q} \dot{u}_{j,i} = \tau_{ij} \dot{u}_{j,i}$$

(A.5)

Consequently,

$$-e_{i,i} = \frac{\partial}{\partial t}(½\rho\dot{u}_i\dot{u}_i) + \partial/\partial t\left(\tau_{ij}u_{j,i}\right) = \dot{E}$$

(A.6)

Note that nothing in this derivation precludes c_{ijpq} or ρ from being functions of position. Consequently, the definition of energy density and energy flux density in Equation (A.1) are correct in heterogeneous and anisotropic elastic media.

An important special case is that of the horizontally-propagating harmonic wave $u_i = U_i(k,\omega,z)\exp(ikx - i\omega t)$, where k is horizontal wavenumber and ω is angular frequency. Such a wave can be a solution to the wave equation only when the medium is vertically-stratified; that is, when c_{ijpq} and ρ depend upon depth, only. When the harmonic wave has the additional property of having zero vertical energy flux, energy density and energy flux density are related by

$$\mathbf{e} = v_x E[1 \quad 0 \quad 0]^T$$

(A.7)

Here, $v_x \equiv \omega/k$ is the horizontal phase velocity. We now prove this result. When motions are confined to the (x,z) plane and are independent of the y-coordinate, $u_y = u_{x,y} = u_{z,y} = 0$. Neither the energy density nor the energy flux density depends upon u_y, as it is zero, or upon τ_{xy}, τ_{zy} or τ_{yy}, as they are all multiplied by quantities that are zero:

$$\tau_{ij}u_{i,j} = \tau_{xy}u_{x,y} + \tau_{yx}u_{y,x} + \tau_{zy}u_{x,y} + \tau_{yz}u_{y,z}\ \ \tau_{yy}u_{y,y} + \text{other terms} = 0 + \text{other terms}$$

$$\dot{u}_i\dot{u}_i = \dot{u}_y\dot{u}_y + \text{other terms} = 0 + \text{other terms}$$

$$e_x = -\tau_{xy}\dot{u}_y + \text{other terms} = 0 + \text{other terms}$$

$$e_z = -\tau_{zy}\dot{u}_y + \text{other terms} = 0 + \text{other terms}$$

(A.8)

The proof makes use of the fact that derivatives of a quantity, say f, with respect to horizontal position x and time t can be performed trivially, as $f_{,x} = ikf$ and $\dot{f} = -i\omega f$. The horizontal component of the energy flux density $e_x = -\tau_{xj}\dot{u}_j$ is then:

$$e_x = i\omega[\tau_{xx}u_x + \tau_{xz}u_z] \quad \text{so} \quad [\tau_{xx}u_x + \tau_{xz}u_z] = \frac{e_x}{i\omega}$$

(A.9)

As the vertical energy flux density is assumed to be zero, $e_z = -\tau_{zj}\dot{u}_j = 0$ and we can write

$$e_z = 0 = i\omega[\tau_{xz}u_x + \tau_{zz}u_z] \quad \text{so} \quad [\tau_{xz}u_x + \tau_{zz}u_z] = 0$$

(A.10)

The energy density equation $E = ½\tau_{ij}u_{i,j} + ½\rho\dot{u}_i\dot{u}_i$ becomes

$$2E = \tau_{xx}u_{x,x} + \tau_{xz}u_{x,z} + \tau_{xz}u_{z,x} + \tau_{zz}u_{z,z} + ½\rho\dot{u}_x\dot{u}_x + ½\rho\dot{u}_z\dot{u}_z \tag{A.11}$$

Performing the x-derivatives yields:

$$2E = ik[\tau_{xx}u_x + \tau_{xz}u_z] + \left[\tau_{xz}u_{x,z} + \tau_{zz}u_{z,z}\right] - \rho\omega^2[u_xu_x + u_zu_z] \tag{A.12}$$

Substituting Eqn. (A.9) yields

$$2E = \frac{k}{\omega}e_x + \left[\tau_{xz}u_{x,z} + \tau_{zz}u_{z,z}\right] - \rho\omega^2[u_xu_x + u_zu_z] \tag{A.13}$$

We now differentiate Eqn. (A.10) with respect to z:

$$[\tau_{xz}u_x + \tau_{zz}u_z]_{,z} = 0 = \left[\tau_{xz}u_{x,z} + \tau_{zz}u_{z,z}\right] + \left[\tau_{xz,z}u_x + \tau_{zz,z}u_z\right]$$
$$\left[\tau_{xz}u_{x,z} + \tau_{zz}u_{z,z}\right] = -\left[\tau_{xz,z}u_x + \tau_{zz,z}u_z\right] \tag{A.14}$$

Substituting this result into Eqn. (A.13) yields

$$2E = \frac{k}{\omega}e_x - \left[\tau_{xz,z}u_x + \tau_{zz,z}u_z\right] - \rho\omega^2[u_xu_x + u_zu_z] \tag{A.15}$$

The equations of motion are $-\rho\omega^2u_x = \tau_{xx,x} + \tau_{xz,z}$ and $-\rho\omega^2u_z = \tau_{xz,x} + \tau_{zz,z}$. Summing them yields:

$$-\rho\omega^2[u_xu_x + u_zu_z] = \left[\tau_{xz,z}u_x + \tau_{zz,z}u_z\right] + \left[\tau_{xx,x}u_x + \tau_{xz,x}u_z\right]$$

or

$$-\left[\tau_{xz,z}u_x + \tau_{zz,z}u_z\right] = \rho\omega^2[u_xu_x + u_zu_z] + \left[\tau_{xx,x}u_x + \tau_{xz,x}u_z\right] \tag{A.16}$$

Substituting this result into Eqn. (A.15) and performing the x derivatives yield

(A.17)

$$2E = \frac{k}{\omega}e_x + ik[\tau_{xx}u_x + \tau_{xz}u_z] \tag{A.18}$$

Substituting Eqn. (A.9) yields

$$2E = \frac{k}{\omega} e_x + ik \frac{e_x}{i\omega} = 2 \frac{k}{\omega} e_x$$

(A.19)

which equals Eqn. (A.7) and competes the proof. Note that the result is correct for any vertically-stratified possibly-anisotropic elastic medium, but requires that the harmonic wave have no vertical energy flux density.

As the formula for energy density E is more complicated than the equation for energy flux density $\mathbf{e}$, Eqn. (A.7) arguably offers no computational advantage. An advantage of computing energy flux directly from Eqn. (A1b) is that one can check that e_z actually is zero.

A.2 Energy Flux Density of a Plane Shear Wave in a Homogeneous, Isotropic Whole Space. We first consider the concept of energy flux density with a simple example of a horizontally propagating, vertically polarized shear wave with $\mathbf{u} = [0,0,U_z]^T \cos(\omega x/\beta - \omega t)$, where $\beta \equiv \sqrt{\mu/\rho}$ is shear velocity and μ is shear modulus. The non-zero components of stress and displacement are:

$$\tau_{xz} = \mu \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \right) = -\omega \rho b U_z \sin(\omega x/\beta - \omega t)$$
$$\dot{u}_z = \omega U_z \sin(\omega x/\beta - \omega t) = \dot{U}_z \sin(\omega x/\beta - \omega t) \quad \text{with} \quad \dot{U}_z^2 \equiv \omega^2 U^2$$

(A.20)

The energy flux density is

$$e_x = -\tau_{xz} \dot{u}_z = 2F \sin^2(\omega x/\beta - \omega t) \quad \text{with} \quad F \equiv \rho \beta \dot{U}_z^2 = \rho^{1/2} \mu^{1/2} \dot{U}_z^2$$

(A.21)

and the time-averaged energy flux density function is F (because a factor of ½ arises from the time-averaging of the sinusoid).

A.3 Conservation of Energy for Seismic Rays. We start be examining the case of the scalar wave equation $\rho \ddot{\varphi} = M \nabla^2 \varphi + \nabla M \cdot \nabla \varphi$ where M is an elastic modulus and both M and ρ are functions of position $\mathbf{x}$, as is the velocity is $c = M^{1/2} \rho^{-1/2}$. This is a scalar analogue to the vector elastic wave equation that shares with it the feature that both density and modulus are spatially variable. Ray theory, which is a high-frequency approximation, the scalar field is parameterized as a series in inverse powers of angular frequency, ω

$$\varphi(\mathbf{x}, t) = \sum_{n=0}^{\infty} (i\omega)^{-n} A_n(\mathbf{x}) \exp\{i\omega(T(\mathbf{x}) - t)\}$$

(A.22)

Here, $T(\mathbf{x})$ is travel time and the $A_n(\mathbf{x})$ are amplitudes (with only A_0 being important at high frequencies). After inserting this parameterization into the wave equation and equating terms of equal powers of ω, the following three equations arise

$$\nabla T = \frac{\mathbf{t}}{c} \quad \text{and} \quad \frac{d\mathbf{t}}{ds} = \mathbf{t} \times [\mathbf{t} \times c^{-1}\nabla c\,]$$

$$\text{and} \quad \frac{\nabla A_0^2}{A_0} \cdot \mathbf{t} = -c\nabla \cdot \left(\frac{\mathbf{t}}{c}\right) - M^{-1}\nabla \mathrm{M} \cdot \mathbf{t}$$

(A.23a,b,c)

Here $\mathbf{t}$ is a unit vector parallel to the ray and s is arc-length along the ray (Fig. A.1). Eqn. (A.23a) indicates that travel time advances in the ray direction at a rate given by the local slowness (reciprocal velocity). The equation (not shown) that results from taking the squared length of Eqn (A.23a) is called the *Eikonal* equation. Equation (A.23b) (the *ray* equation) provides a method for calculating ray paths and Eqn. (A.23c) (the *transport* equation) provides a method for calculating amplitudes.

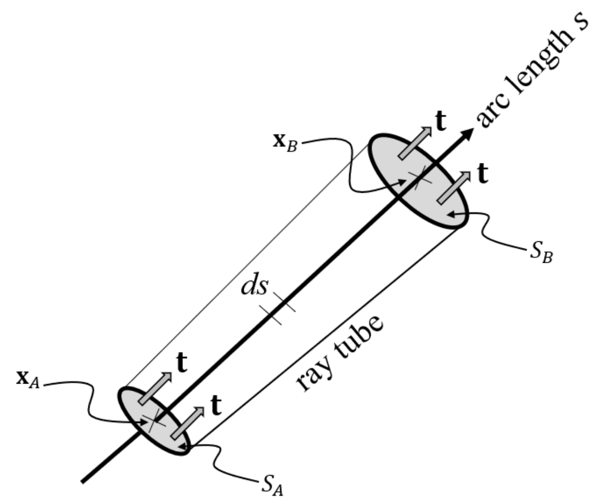

Fig. A.1. Ray with arc length s propagating in the $\mathbf{t}$ direction, surrounded by a ray tube with cross-sectional area S_A at position $\mathbf{x}_A$ and area S_B at position $\mathbf{x}_B$. This ray is defocusing, as $S_B > S_A$.

We now show that the transport equation implies conservation of an energy flux density scalar F. We note that both ray and transport equations involve only relative changes is velocity and modulus, and so are insensitive to absolute size of those quantities. Consequently, ray theory can constrain energy flux density only up to a multiplicative constant. Motivated by our result from the plane shear wave (Eqn. A21), we write the amplitude as $A_0^2 = cM^{-1}Ff_0^{-1}$, where F is energy flux density scalar and f_0 is a constant. The transport equation becomes

$$\frac{\nabla\left(cM^{-1}Ff_0^{-1}\right)}{cM^{-1}Ff_0^{-1}} \cdot \mathbf{t} = -c\ \nabla \cdot \left(\frac{\mathbf{t}}{c}\right) - cM^{-1}\nabla \mathrm{M} \cdot \left(\frac{\mathbf{t}}{c}\right)$$

(A24)

Applying the chain rule yields

$$\left(\frac{\nabla F}{F}\right)\cdot \mathbf{t} + c^{-1}\nabla c \cdot \mathbf{t} + M\nabla M^{-1}\cdot \mathbf{t} = -\nabla \cdot \mathbf{t} + c^{-1}\nabla c \cdot \mathbf{t} + M\nabla M^{-1}\cdot \mathbf{t}$$

(A25)

Here, we have used the identity $B\nabla B^{-1} = -B^{-1}\nabla B$. After canceling two pairs of terms, the equation becomes

$$\frac{\nabla F}{F}\cdot \mathbf{t} = -\nabla \cdot \mathbf{t}$$

(A26)

This equation implies that the energy flux density $\mathbf{e} \equiv F\mathbf{t}$ is conserved; that is:

$$\nabla \cdot \mathbf{e} = \nabla \cdot (F\mathbf{t}) = \nabla F \cdot \mathbf{t} + F\nabla \cdot \mathbf{t} = 0$$

(A.27)

As was hypothesized, energy flux density function is $F = f_0 c^{-1} M A_0^2 = f_0 \rho^{½} M^{½} A_0^2$, with the constant f_0 undetermined.

In ray theory, energy propagates parallel to the ray direction and is conserved. Furthermore, the energy density E has no time dependence, as the conservation equation $\partial E/\partial t + \nabla \cdot \mathbf{e} = 0$ reduces to $\partial E/\partial t = 0$. There is no sense of energy *storage* in ray theory.

Continuing the shear wave example, we consider an isotropic elastic medium with shear modulus $\mu(x,y)$ and density $\rho(x,y)$ (that is, no z-dependence) and a horizontally-propagating, vertically polarized shear wave. The wave equation $\rho\ddot{u}_z = \tau_{zx,x} + \tau_{zy,y}$ has no $\tau_{zz,z}$ term because no quantity varies with z. The stress-displacement relationships are $\tau_{zx} = \mu u_{z,x}$ and $\tau_{zy} = \mu u_{z,y}$. Differentiating them and inserting them into the wave equation yields $\rho\ddot{u}_z = \mu\nabla^2 u_z + \nabla\mu \cdot \nabla u_z$ (where ∇ is two-dimensional). Equating $\varphi = u_z$, $M = \mu$ and $A_0 = U_z$ we obtain

$$F = f_0 \rho^{½} \mu_0^{½} U_z^2$$

(A.28)

This result matches the result from wave theory (Eqn. A.21) after the undetermined constant is set to $f_0 = \omega^2$.

The conservation equation $\nabla \cdot (F\mathbf{t}) = 0$ has a simple interpretation in terms of the change in cross-sectional area S of the ray tube formed from a group of neighboring rays (Fig. A.1). Integrating the conservation equation over the volume of the ray tube and applying Gauss' theorem yields

$$0 = \iiint_{\text{ray tube}} \nabla \cdot (F\mathbf{t})\, \mathrm{d}V = \iint_{S_A} F\mathbf{t}\cdot(-\mathbf{t})\, \mathrm{d}S + \iint_{S_B} F\mathbf{t}\cdot\mathbf{t}\, \mathrm{d}S \approx -S_A F_A + S_B F_B$$

(A.29)

Here we have used the fact that the surface integral involving the conical side of tube is zero, as its surface normal is perpendicular to the ray direction, and have assumed that the ray tube is sufficiently thin that F is approximately constant over each of the end caps (with values F_A and F_B, respectively). Thus,

$$\frac{F_B}{F_A} = \frac{S_A}{S_B}$$

(A.30)

That is, the flux F varies in inverse proportion to the cross-sectional area S of the ray tube (a behavior called *geometrical spreading*).

The amplitude $A = (F f_0^{-1} \rho^{-1/2} M^{-1/2})^{1/2}$ can change either because the energy flux density F changes as rays converse/diverge or because material parameters, say ρ and M, are varied. Giving the material parameters the generic names m_1 and m_2, we can write

$$dA = \left.\frac{\partial A}{\partial F}\right|_{m_1,m_1} \mathrm{d}F + \left.\frac{\partial A}{\partial m_1}\right|_{F,m_2} \mathrm{d}m_1 + \left.\frac{\partial A}{\partial m_2}\right|_{F,m_1} \mathrm{d}m_2$$

(A.31)

To first order, each effect operates independently.

A.4 Amplitude Sensitivity. The amplitude sensitivity is defined as the fractional change in amplitude with respect to changes in a material property, at constant energy flux density. In order for the sensitivity to be well-defined, both the material property that is being varied and the one being held constant needs to be specified. In our notation, notation, $s_{m_1|m_2} \equiv A^{-1} \partial A / \partial m_1|_{m_2}$ is the is the sensitivity to changes in material property m_1 at constant m_2.

We first consider the case of a plane shear wave as it propagates from one region to another of constant density ρ_0 but varying rigidity μ. If reflections and conversions at boundaries are neglected, so that the flux $F = \rho_0^{1/2} \mu^{1/2} \dot{U}_z^2$ is constant, then

$$dF = 0 = \tfrac{1}{2} \rho_0^{1/2} \mu^{-1/2} \dot{U}^2 d\mu + 2 \rho_0^{1/2} \mu^{1/2} \dot{U} d\dot{U}$$

(A.32)

Hence the sensitivity to rigidity (at constant density) is

$$s_{\mu|\rho} \equiv \frac{1}{\dot{U}} \frac{d\dot{U}}{d\mu} = -\frac{\tfrac{1}{4} \rho_0^{1/2} \mu^{-1/2} \dot{U}}{\rho_0^{1/2} \mu^{1/2} \dot{U}} = -\frac{1}{4\mu}$$

(A.33)

As $\mu = \beta^2 \rho_0$, we can use the derivative $d\mu/d\beta = 2\beta\rho_0 = (2\beta^2\rho_0)/\beta = 2\mu/\beta$ to convert to an expression involving shear velocity:

$$s_{\mu|\rho} \equiv \frac{1}{\dot{U}} \frac{d\dot{U}}{d\beta} = \frac{1}{\dot{U}} \frac{d\dot{U}}{d\mu} \frac{d\mu}{d\beta} = -\frac{1}{4\mu} \frac{2\mu}{\beta} = -\frac{1}{2\beta}$$

(A.34)

So, as velocity increases, amplitude decreases. Similarly, at constant energy flux density and with constant μ_0 and variable ρ

$$dF = 0 = ½\rho^{-½}\mu_0^{½}\dot{U}^2 d\rho + 2\rho^{½}\mu_0^{½}\dot{U}d\dot{U} \tag{A.35}$$

So, the sensitivity to density (at constant rigidity) is:

$$s_{\rho|\mu} \equiv \frac{1}{\dot{U}}\frac{d\dot{U}}{d\rho} = -\frac{½\rho^{-½}\mu_0^{½}\dot{U}}{2\rho^{½}\mu_0^{½}\dot{U}} = -\frac{1}{4\rho} \tag{A.36}$$

As $\rho = \mu_0\beta^{-2}$, we can use the derivative $d\rho/d\beta = -2\mu_0\beta^{-3} = -(2\beta^{-2}\mu_0)/\beta = -2\rho/\beta$ to convert to an expression involving shear velocity:

$$s_{\rho|\mu} \equiv \frac{1}{\dot{U}}\frac{d\dot{U}}{d\beta} = \frac{1}{\dot{U}}\frac{d\dot{U}}{d\rho}\frac{d\rho}{d\beta} = \left(-\frac{1}{4\rho}\right)\left(-\frac{2\rho}{\beta}\right) = \frac{1}{2\beta} \tag{A.37}$$

So, as velocity increases, amplitude *increases*.

A.5 Ratio of Amplitude to Square Root of Flux. The ratio R of the amplitude to the square root of energy flux density is a useful quantity because it depends only on material properties. For a plane shear wave with amplitude $\dot{U}$ and energy flux density scaler $F = ½\rho^{½}\mu^{½}\dot{U}^2$ (Eqn. A.21) this ratio is:

$$R \equiv \frac{\dot{U}}{\sqrt{F}} = (\rho\beta)^{-½} = (\rho\mu)^{-¼} \tag{A.38}$$

When a plane shear wave passes from medium A to medium B, and if reflections and conversions are neglected so that the flux is constant, then

$$F = \rho_A\beta_A\dot{U}_A^2 = \rho_B\beta_B\dot{U}_B^2$$

$$\frac{R_B}{R_A} = \frac{\dot{U}_B}{\dot{U}_A} = \left(\frac{\rho_A\beta_A}{\rho_B\beta_B}\right)^{½} = \left(\frac{\rho_A\mu_A}{\rho_B\mu_B}\right)^{¼} \tag{A.39}$$

Suppose that the wave propagated from a reference region, where it has ratio R_{ref} into another, where it has ratio R. We can write:

$$R_{ref} \equiv \frac{\dot{U}_{ref}}{\sqrt{F_{ref}}} \quad \text{and} \quad R \equiv \frac{\dot{U}}{\sqrt{F}} \quad \text{and} \quad \dot{U}_{ref} = R_{ref}\sqrt{F_{ref}} \quad \text{and} \quad \dot{U} = R\sqrt{F}$$

$$\log\dot{U}_{ref} = \log R_{ref} + ½\log F_{ref} \quad \text{and} \quad \log\dot{U} = \log R + ½\log F$$

(A.40)

If the flux of the wave is equal in these two regions, $F_{ref} = F$, and

$$\frac{1}{\dot{U}}\Delta\dot{U} = \Delta\log\dot{U} = \log R_B - \log R_A = \log(R_B/R_A)$$

(A.41)

Given a change in material property, say $\Delta m \equiv m - m_{ref}$, the amplitude sensitivity can be approximated using the finite difference derivative

$$s \equiv \frac{1}{\dot{U}}\frac{d\dot{U}}{dm} \approx \frac{1}{\dot{U}}\frac{\Delta\dot{U}}{\Delta m} = \frac{\Delta\log\dot{U}}{\Delta m} = \frac{\log R - \log R_{ref}}{\Delta m} = \frac{\log\left(R/R_{ref}\right)}{\Delta m}$$

(A.42)

A.6 Total Energy Flux Density of a Rayleigh Wave. We consider a plane Rayleigh wave propagating in the x-direction in an isotropic medium with vertically stratified material properties, e.g. $\rho(z)$, where z is depth. Our goal is to compute the ratio R of the vertical displacement to the square root of the total horizontal energy flux. The Rayleigh wave motions can be expressed in the frequency-horizontal wavenumber domain, using the displacement-stress vector approach, with vector

$$\mathbf{d}(\omega, k, z) \equiv [U_x \quad -iU_z \quad T_{xz} \quad -iT_{zz}\,]^T$$

(A.43)

Here, U_x and U_z are displacement amplitudes, T_{xz} and T_{zz} are stress amplitudes, ω is angular frequency and k_x is horizontal wavenumber. We calculate energy flux density using the approach of Synge (1956-1957) and Menke and Rhoads (2023). The displacement associated with the positive and negative frequencies is

$$u_x = 2U_x^R\cos(\omega t) + 2U_x^I\sin(\omega t)$$
$$u_z = 2U_z^R\cos(\omega t) + 2U_z^I\sin(\omega t)$$

(A.44)

Here, superscripts R and I refer to real and imaginary parts, respectively. The squared amplitude is

$$u_x^2 = 4(U_x^R)^2\cos^2(\omega t) + 4(U_x^I)^2\sin^2(\omega t) + 4U_x^R U_x^I\cos(\omega t)\sin(\omega t)$$

(A.45)

An analogous expression holds for u_z^2. The time-averaged values are

$$\langle u_x^2\rangle = 2(U_x^R)^2 + 2(U_x^I)^2 \quad \text{and} \quad \langle u_z^2\rangle = 2(U_z^R)^2 + 2(U_z^I)^2$$

(A.46)

and the root-mean-squared amplitudes are

$$u_x^{rms} = [2(U_x^R)^2 + 2(U_x^I)^2]^{½} \quad \text{and} \quad u_z^{rms} = [2(U_z^R)^2 + 2(U_z^I)^2]^{½}$$

(A.47)

The time averaged energy flux density of the positive and negative frequencies is calculated in a similar wave and is found to me

$$e_i = -2\omega\left(T_{ij}^R U_j^I - T_{ij}^I U_j^R\right)$$

(A.48)

The energy flux density scalar is the energy flux density in the x-direction is

$$F = e_x = -2\omega(T_{xx}^R U_x^I + T_{xz}^R U_z^I - T_{xx}^I U_x^R - T_{xz}^I U_z^R)$$

(A.49)

and the total, vertically integrated horizontal flux is

$$F_T = \int_0^\infty F \, \mathrm{d}z$$

(A.50)

The displacement-to-square-root-of-flux ratio R and sensitivity s can then be defined as

$$R \equiv \frac{u_z^{rms}}{\sqrt{F_T}} \quad \text{and} \quad s \equiv \frac{1}{u_z^{rms}} \frac{\partial u_z^{rms}}{\partial v_r}\bigg|_{z=0}$$

(A.51)

Here v_r is the phase velocity of the Rayleigh wave.

Although the motion-stress vector does not include an explicit entry for T_{xx}, it can be calculated from known quantities, starting with the definition of stress

$$\tau_{11} = (\lambda + 2\mu)\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_3}{\partial x_3} \quad \text{and} \quad \tau_{33} = \lambda\frac{\partial u_1}{\partial x_1} + (\lambda + 2\mu)\frac{\partial u_3}{\partial x_3}$$

(A.52)

We first note that $\partial u_1/\partial x_1 = i\omega p U_1$, where $p \equiv k_x/\omega$ is horizontal slowness, so it can be calculated from $\mathbf{d}$. Solving the τ_{33} for $\partial u_3/\partial x_3$, substituting the result into the τ_{11} equation, and rearranging yields

$$\tau_{11} = \lambda(\lambda + 2\mu)^{-1}\tau_{33} + [(\lambda + 2\mu) - \lambda^2(\lambda + 2\mu)^{-1}]\frac{\partial u_1}{\partial x_1}$$

(A.53)

A.7 Perturbative Formula for Amplitude Sensitivity The formula for sensitivity (Eqn. A.42) is based on a direct (or finite-difference) approach, in the sense that both numerator and denominator in the ratio R/R_{ref} are computed directly using the propagator matrix method. Such an approach is cumbersome in geo-tomography, as R needs to be computed numerous times. Here we use perturbation theory to derive a formula for R that only involves quantities that can be computed from the reference model, together with the asserted vertically-stratified perturbations in elastic moduli $\delta\lambda$ and $\delta\mu$ and density $\delta\rho$. Key to our derivation is the calculation of the change in the displacement wave function $\mathbf{u}(\omega, z)$ arising from the perturbation in Earth structure.

Consider a dispersion curve $\omega(k)$ (Fig. A.2, black curve) that is perturbed to $\omega(k) + \Delta\omega(k)$ (red curve) by a "fast" perturbation in Earth structure. At constant frequency, a reference point $\left(k_{ref}, \omega_{ref}\right)$ (black circle) on the unperturbed curve moves to the point $\left(k_p, \omega_{ref}\right)$ (red circle) on the perturbed curve. The wavenumber is reduced and the phase velocity is increased. This point has the same wavenumber, but higher frequency and velocity, than the point $\left(k_p, \omega_a\right)$ (grey circle) on the unperturbed dispersion curve.

The displacement wave function $\mathbf{u}(\omega, z)$ and its stress $\boldsymbol{\tau}(\omega, z)$ differs between points (ref) and *(a)*, but as they are both points on the unperturbed curves, they are known by assumption. Consequently, R^{ref} and R^a are known, too. However, rather than to tabulate R^a along the curve, we use a first-order approximation to calculate R^a from R^{ref} and its derivative with respect to wavenumber. In contrast, the wavefunctions at points (a) and (p) are equal, at least if one ignores the contribution of higher modes (overtones), as only modes with the same wavenumber couple for a vertically-stratified perturbation in structure (owing to the orthogonality of the horizontal wavefunctions).

The following quantities can be calculated along the unperturbed dispersion curve: the phase velocity $c(\omega) = \omega/k(\omega)$; the vertically-integrated energy flux $F(\omega)$; the quantity $R_{ref}(\omega) = u_3^{rms,ref}/\sqrt{F_{ref}}$ where $u_3^{rms,ref}(\omega)$ is measured at $z = 0$; and the energy integrand

$$G(z, \omega) = \frac{[u_1^{rms}(z)]^2 + [u_3^{rms}(z)]^2}{[u_3^{rms}(z = 0)]^2} \tag{A.54}$$

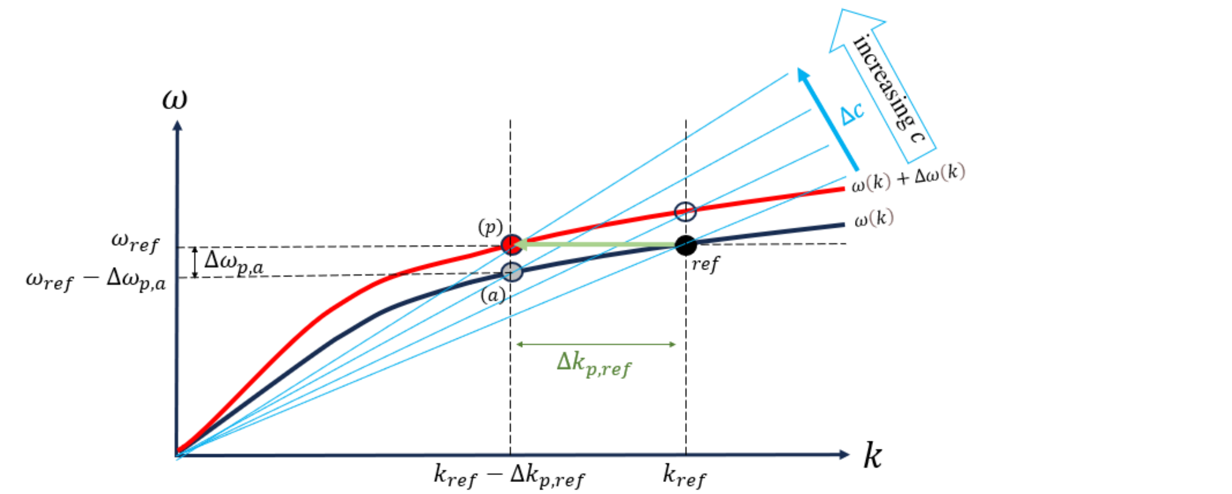

Fig. A.2. Unperturbed dispersion curve $\omega(k)$ (black curve) and dispersion curve $\omega(k) + \Delta\omega(k)$ (red curve) after a perturbation in Earth structure. Lines of constant phase velocity $c = \omega/k$ (blue lines) are also shown. A reference point (k_{ref}, ω_{ref}) (black circle) on the unperturbed dispersion curve has the same frequency as the point (k_p, ω_{ref}) (red circle) on the perturbed dispersion curve. The point $(k_p, \omega_{ref} - \Delta\omega)$ on the unperturbed curve (grey circle) with wavenumber k_p has frequency less by the amount $\Delta\omega_{p,a}$.

Using finite differences, the derivative $dR/d\omega$ and the group velocity $dk/d\omega$ can also be calculated. Then, using the chain rule

$$\frac{dR}{dk} = \left(\frac{dR}{d\omega}\right)\bigg/\left(\frac{dk}{d\omega}\right)$$

(A.55)

Now consider a vertically-stratified perturbation in material properties that shifts the dispersion curve to (Fig. A.2, red curve)

$$c_p = c_{ref} + \Delta c_{p,ref}$$

(A.56)

We assume that $\Delta c_{p,ref}$ has been calculated from $(\delta\lambda, \delta\mu, \delta\rho)$ and $\mathbf{u}^{ref}(\omega, z)$ and $\boldsymbol{\tau}^{ref}(\omega, z)$ using the well-known perturbative formula (e.g. Aki and Richards, 2009, their Section 7.3.2).

At constant frequency ω_{ref}, the shift in wavenumber is

$$\Delta k_{p,ref} = k_p - k_{ref} = \omega_{ref}\left[c_{ref} + \Delta c_{p,ref}\right]^{-1} - \frac{\omega_{ref}}{c_{ref}} =$$

$$\frac{\omega_{ref}}{c_{ref}}\left[1 + \frac{\Delta c_{p,ref}}{c_{ref}}\right]^{-1} - \frac{\omega_{ref}}{c_{ref}} = \frac{\omega_{ref}}{c_{ref}}\left[1 - \frac{\Delta c_{p,ref}}{c_{ref}}\right] - \frac{\omega_{ref}}{c_{ref}} = -\frac{\omega_{ref}\Delta c_{p,ref}}{c_{ref}^2}$$

(A.57)

or

$$\frac{\Delta k_{p,ref}}{k_{ref}} = -\frac{\Delta c_{p,ref}}{c_{ref}}$$

(A.58)

At wavenumber k_a on the unperturbed dispersion curve, the ratio R is to first order

$$R_a = R_{ref} + \left.\frac{dR}{dk}\right|_{k_{ref}} \Delta k_{a,ref} \equiv R_{ref} + \Delta R_{a,ref}$$

(A.59)

In order to calculate R_p from R_a, we use the rule $F = Ec$, where E is the vertically-integrated energy density, together with the fact that time-averaged kinetic and potential energies of a mode are equal:

$$R = \frac{u_3^{rms}(z=0)}{\sqrt{Ec}} = u_3^{rms}(z=0)\left(c\int_0^\infty \omega^2 \rho\{[u_1^{rms}(z)]^2 + [u_3^{rms}(z)]^2\}\,dz\right)^{-½} = c^{-½}\omega^{-1}I^{-½}$$

$$\text{with} \quad I \equiv \int_0^\infty \rho \frac{[u_1^{rms}(z)]^2 + [u_3^{rms}(z)]^2}{[u_3^{rms}(z=0)]^2}\,dz$$

(A.60)

Then, R_{ref} is calculated by first perturbing R_{ref} to R_a via Eqn. (5) and then perturbing to R_a to R_p by replacing c_a with c_p, ω_a with ω_p and I_a with I_p:

$$R_p = \left(R_{ref} + \frac{dR}{dk}\Delta k_{p,ref}\right)\left(\frac{c_p}{c_a}\right)^{-½}\left(\frac{\omega_p}{\omega_a}\right)^{-1}\left(\frac{I_p}{I_a}\right)^{-½}$$

(A.61)

As points (p) and (a) have the same wavenumber, $\omega_p/\omega_a = c_p/c_a$. As no mode mixing occurs, $u_i^p = u_i^a$ and

$$I_p = I_a + \Delta I_{p,a} = \int_0^\infty \rho_{ref}\, G_a(z)\,dz + \int_0^\infty \delta\rho\, G_a(z)\,dz =$$

$$I_{ref} + \left(\int_0^\infty \rho_{ref} \left.\frac{dG}{dk}\right|_{ref} dz\right)\Delta k_{a,ref} + \left(\int_0^\infty \delta\rho\, G_{ref}(z)\,dz\right) =$$

$$\equiv I_{ref} + J_{ref}\Delta k_{p,ref} + K_{ref}$$

(A.62)

The ratio R_p is then

$$R_p = \left(R_{ref} + \frac{dR}{dk}\Delta k_{a,ref}\right)\left(\frac{c_p}{c_a}\right)^{-3/2}\left(\frac{I_p}{I_a}\right)^{-½} =$$

$$R_{ref}\left(1+\frac{1}{R_{ref}}\frac{dR}{dk}\Delta k_{p,ref}\right)\left(c_{ref}+\Delta c_{p,ref}\right)^{-\frac{3}{2}}\left(c_{ref}+\frac{dc}{dk}\Delta k_{p,ref}\right)^{\frac{3}{2}}\times$$
$$\left(I_{ref}+J_{ref}\Delta k_{p,ref}+K_{ref}\Delta k_{p,ref}\right)^{-½}\left(I_{ref}+J_{ref}\Delta k_{p,ref}\right)^{½}=$$
$$R_{ref}\left(1+\frac{1}{R_{ref}}\frac{dR}{dk}\Delta k_{p,ref}-\frac{3}{2}\frac{\Delta c_{p,ref}}{c_{ref}}+\frac{3}{2}\frac{1}{c_{ref}}\frac{dc}{dk}\Delta k_{p,ref}-\frac{1}{2}\frac{K_{ref}}{I_{ref}}\right)$$

(A.63)

We can write this result succinctly as

$$R_p/R_{ref}=(1+T_1-T_2+T_3-T_4)$$

(A.64)

where the Ts correspond to the terms of Eqn. (10). This result is valid irrespective of whether the medium is isotropic or anisotropic, with the caveat that for anisotropic models $\Delta c_{p,ref}$ must be properly computed. Term T_1 is strongly frequency dependent. Terms T_2 and T_3 are unequal, as they represent points on different dispersion curves, and $(-T_2+T_3)$ is very weakly frequency-dependent. Term T_4 depend on density, only, and is zero when $\delta\rho=0$, implying that the behavior of R_p/R_{ref} depends on whether the elastic moduli or density is being perturbed.

As a validation test, we use Eqn. (A.63) to compute R/R_{ref} for a simple layer-over-a-half-space Earth model (the same Case 1, described in a subsequent section), alternately with a 5% increase in elastic moduli or 5% decrease in density. The prediction agrees well with those of the direct method (Fig. A.3).

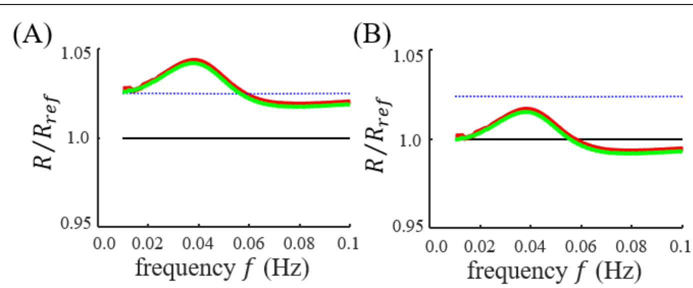

Fig. A.3. Comparison of direct and perturbative calculation of the ratio R/R_{ref}, for the reference Earth model of Case 1. (A) The elastic moduli are perturbed by 5% and the density is held constant. (B) The elastic moduli are held constant and the density is perturbed by 5%. In both cases, the direct (red curves) perturbative (green curves) calculations match well. Omitting all terms except T_2 and T_3 leads to a frequency-independent result (blue curves).