Derivative of the Surface Wave Wave-function with Respect to Horizontal Wavenumber

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Previous work (Menke Research Note 284, 2014, Sec. A.7,), concerning the amplitude sensitivity of surface waves, defines the quantity

$$R \equiv \frac{u_i}{\sqrt{F_T}} = \frac{u_i}{\sqrt{Ec}} = \frac{u_i}{\sqrt{E}} c^{-½}$$

(1)

where u_i is the displacement wave-function, F_T is the vertically-integrated horizontal energy flux density, E is the vertically-integrated energy density, $c \equiv \omega/k_x$ is the phase velocity, ω is angular frequency, and k_x is horizontal wavenumber. Previously, I computed the derivative dR/dk_x along the dispersion curve $\omega(k_x)$ by finite differences. Here, I present an alternate method, based on solving a differential equation.

The vertically integrated energy density is

$$E = \omega^2 I \quad \text{with} \quad I \equiv \langle u_i^* u_i \rangle = \int_0^\infty \rho u_i^* u_i dz$$

(2)

Suppose now that u_i has been normalized so that $I = 1$. Then, $E = \omega^2$, $F_T = \omega^2 c$ and $R = \omega^{-1} c^{-½} u_i$. The derivative is

$$\frac{dR}{dk_x} = \omega^{-1} c^{-½} \frac{du_i}{dk_x} - \omega^{-2} c^{-½} u_i \frac{d\omega}{dk_x} - ½ u_i c^{-3/2} \frac{dc}{dk_x}$$

(3)

The derivative of the phase velocity is

$$\frac{dc}{dk_x} = \frac{d}{dk_x}(\omega k_x^{-1}) = k_x^{-1} \frac{d\omega}{dk_x} - \omega k_x^{-2}$$

(4)

Consequently, for a specified point $\left(k_x^{ref}, \omega^{ref}\right)$ on the dispersion curve, all quantities, except du_i/dk_x, appearing in Eqn. (3) can be calculated by known methods.

We now show how to compute $g_i \equiv du_i/dk_x$ (with the understanding that u_i is normalized so that that $I = 1$).

Part 1. General method

Suppose that the displacement satisfies the homogenous linear differential equation,

$$\mathcal{L}_{pi}(z, k_x, \omega)\ u_i(z, k_x, \omega) = 0 \tag{5}$$

With homogenous boundary conditions

$$\mathcal{B}_{pi}^{(0)}(z = 0\ k_x, \omega) = 0 \quad \text{and} \quad \mathcal{B}_{pi}^{(\infty)}(z \to \infty\ k_x, \omega) = 0. \tag{6}$$

Here, $\mathcal{L}_{pi}$, $\mathcal{B}_{pi}^{(0)}$ and $\mathcal{B}_{pi}^{(\infty)}$ are linear differential operators. Taken together, Eqns. (5-6) are an eigenvalue problem that defines the dispersion relation $\omega(k_x)$ and the wave-function $u_i(z, k_x)$ (up to a multiplicative constant). Subsequently, we shall drop ω from the variable lists as it is now considered a function of k_x.

We differentiate Eqns. (5) with respect to wavenumber (along the dispersion curve), to find

$$\mathcal{L}_{pi}(z, k_x)\ g_i\ = f_p \ \text{ with } \ f_p \equiv -\left(\frac{d}{dk_x}\mathcal{L}_{pi}(z, k_x)\right)u_i(z, k_x) \tag{7}$$

Eqn. (7) an inhomogeneous equation for the derivative $g_i \equiv du_i/dk_x$. Note that it has the same differential operator as the equation for u_i (Eqn. 5). To first order

$$u_i(z, k_x) = u_i\left(z, k_x^{ref}\right) + g_i\left(z, k_x^{ref}\right) \Delta k_x \quad \text{with } \ \Delta k_x \equiv k_x - k_x^{ref} \tag{8}$$

Applying the boundary conditions yields

$$\begin{aligned} \mathcal{B}_{pi}^{(0)}\ u_i(z, k_x) = 0 = \mathcal{B}_{pi}^{(0)} u_i\left(z, k_x^{ref}\right) + \mathcal{B}_{pi}^{(0)} g_i\left(z, k_x^{ref}\right) \Delta k_x \\ \mathcal{B}_{pi}^{(\infty)}\ u_i(z, k_x) = 0 = \mathcal{B}_{pi}^{(\infty)} u_i\left(z, k_x^{ref}\right) + \mathcal{B}_{pi}^{(\infty)} g_i\left(z, k_x^{ref}\right) \Delta k_x \end{aligned} \tag{9}$$

As the boundary conditions on u_i must be satisfied irrespective of Δk_x, $\mathcal{B}_{pi}^{(0)} g_i = \mathcal{B}_{pi}^{(\infty)} g_i = 0$; that is, g_i must satisfy the same boundary conditions at $z = 0$ and $z \to \infty$ as does u_i. As g_i and u_i have the same differential operator and same boundary conditions, $g_i^H = u_i$ is a solution to the homogeneous version of Eqn. (7).

Eqn. (7) can be solved numerically discretizing $\mathcal{L}_{pi}\ g_i\ = f_p$ into the matrix equation $\mathbf{L}\ \mathbf{g} = \mathbf{f}$ (say, with increment Δz) and solving it with standard linear algebraic methods. Suppose that a particular solution g_i^{par} has been constructed. The general solution is

$$g_i = g_i^{par} + \alpha u_i$$

(10)

The parameter α is determined by the requirement that $u_i(z, k_x)$ in Eqn. (8) has $I = 1$ irrespective of the value of Δk_x:

$$I = 1 = \langle \{u_i^* + [\{g_i^{par*} + \alpha u_i^*\}]\Delta k_x\}\{u_i + [g_i^{par} + \alpha u_i]\,\Delta k_x\}\rangle \approx$$

$$\{\langle u_i^* u_i\rangle + [\langle u_i^* g_i^{par}\rangle + \alpha\langle u_i^* u_i\rangle]\,\Delta k_x\} + \{[\langle u_i g_i^{par*}\rangle + \alpha\langle u_i^* u_i\rangle]\,\Delta k_x\} =$$

$$1 + [\langle u_i^* g_i^{par}\rangle + \alpha]\,\Delta k_x + [\langle u_i g_i^{par*}\rangle + \alpha]\,\Delta k_x =$$

$$1 + [\langle u_i^* g_i^{par}\rangle + \langle u_i g_i^{par*}\rangle + 2\alpha]\,\Delta k_x$$

$$1 + [2\,\mathrm{real}\ \langle u_i^* g_i^{par}\rangle + 2\alpha]\,\Delta k_x$$

(11)

Consequently,

$$\alpha = -\,\mathrm{real}\ \langle u_i^* g_i^{par}\rangle$$

(12)

Part 2, Example of Acoustic Surface Waves.

Consider the simplified case of an acoustic surface wave propagating in layer over a half-space, with the wave-function corresponding to pressure $p(z)$, where z is depth. The equation of motion is (Menke and Abbott, 1989, Eqn. 8.4.4):

$$p(z) = \frac{k_x^2}{\omega^2}\frac{\lambda(z)}{\rho(z)}p(z) - \frac{\lambda(z)}{\omega^2}\frac{d}{dz}\frac{1}{\rho(z)}\frac{d}{dz}p(z)$$

(13)

where λ is incompressibility, ρ is density and ω is angular frequency. This equation can be manipulated into the equivalent form

$$\left\{\left(\frac{\omega^2}{c^2} - k_x^2\right) + \frac{d^2}{dz^2} - \frac{1}{\rho}\frac{d\rho}{dz}\frac{d}{dz}\right\}p(z) = 0 \text{ or}$$

$$\mathcal{L}\,p(z) = 0$$

(14)

Here, $c \equiv \sqrt{\lambda/\rho}$ is the local acoustic velocity, and $\mathcal{L}$ is shorthand for the differential operator in the braces. This equation, together with the boundary conditions $p(z = 0) = 0$ and $p(z \to \infty) = 0$ can be satisfied only by certain combinations of (k_x, ω). For a layer of thickness H and material properties (λ_1, ρ_1) over a half-space of material properties (λ_2, ρ_1), and with $c_1 < c_2$, the dispersion function $k_x(\omega)$ satisfies a known transcendental equation of the form $D(k_x, \omega) = 0$ (Menke and Abbott, 1989, Eqn. 8.5.9) that is easy to solve numerically. Furthermore, the vertical wave-function is known analytically, being sinusoidal in the layer and exponentially-decaying in the half-space. Conventionally, the vertical wave-function is normalized so that

$$I \equiv \langle p^2 \rangle \equiv \int_0^\infty \frac{p^2}{\lambda} dz = 1$$

(15)

A differential equation for the derivative $g(z) \equiv dp/dk_x$ can be found by differentiating Eqn. (14):

$$\left\{ \left(\frac{\omega^2}{c^2} - k_x^2 \right) + \frac{d^2}{dz^2} - \frac{1}{\rho} \frac{d\rho}{dz} \frac{d}{dz} \right\} g(z) = -2 \left(\frac{\omega}{c^2} \frac{d\omega}{dk_x} - k_x \right) p(z)$$

or

$$\mathcal{L}\, g(z) = f(z)$$

(15)

Note that $d\omega/dk_x$ is the group velocity. The boundary conditions on $p(z)$ are independent of horizontal wavenumber, implying that $g(z=0)=0$ and $g(z \to 0) = 0$. Note that $p(z)$ and $g(z)$ have the same differential operator and boundary conditions. Consequently, the homogeneous solution to Eqn. (15) is $g_H \equiv p$. Eqn. (15) can be solved numerically for an particular solution g^{par} by discretizing $\mathcal{L}\, g(z) = f(z)$ into the matrix equation $\mathbf{L}\,\mathbf{g} = \mathbf{f}$, say with increment Δz) and solving it with standard linear algebraic methods. The general solution is then

$$g(z) = g^{par} + \alpha g_H$$

(16)

where α is an as-yet-undetermined parameter.

Given the wave-function $p\left(z, k_x^{ref}\right)$ for a particular reference wavenumber k_x^{ref}, the wave-function at a neighboring wavenumber $k_x^{ref} + \Delta k_x$ is

$$p\left(z, k_x^{ref} + \Delta k_x\right) \approx p\left(z, k_x^{ref}\right) + g\left(z, k_x^{ref}\right) \Delta k_x$$

or

$$p\left(z, k_x^{ref} + \Delta k_x\right) \approx p\left(z, k_x^{ref}\right) + \left(g^{par}\left(z, k_x^{ref}\right) + \alpha g_H\left(z, k_x^{ref}\right) \right) \Delta k_x$$

(17)

The parameter α is determined by the condition that $I = 1$ irrespectice of the value of Δk_x:

$$I = 1 = \left\langle \left(p\left(z, k_x^{ref}\right) + \left(g^{par}\left(z, k_x^{ref}\right) + \alpha g_H\left(z, k_x^{ref}\right) \right) \Delta k_x \right)^2 \right\rangle$$

(18)

After identifying $g_H(z, k_x^{ref}) = p(z, k_x^{ref})$ and assuming that $\langle\left(p^2(z, k_x^{ref})\right)\rangle = 1$, Eqn. (7) becomes

$$1 + \{\langle g^{par}(z, k_x^{ref})p(z, k_x^{ref})\rangle + \alpha\}\Delta k_x \approx 1$$

(19)

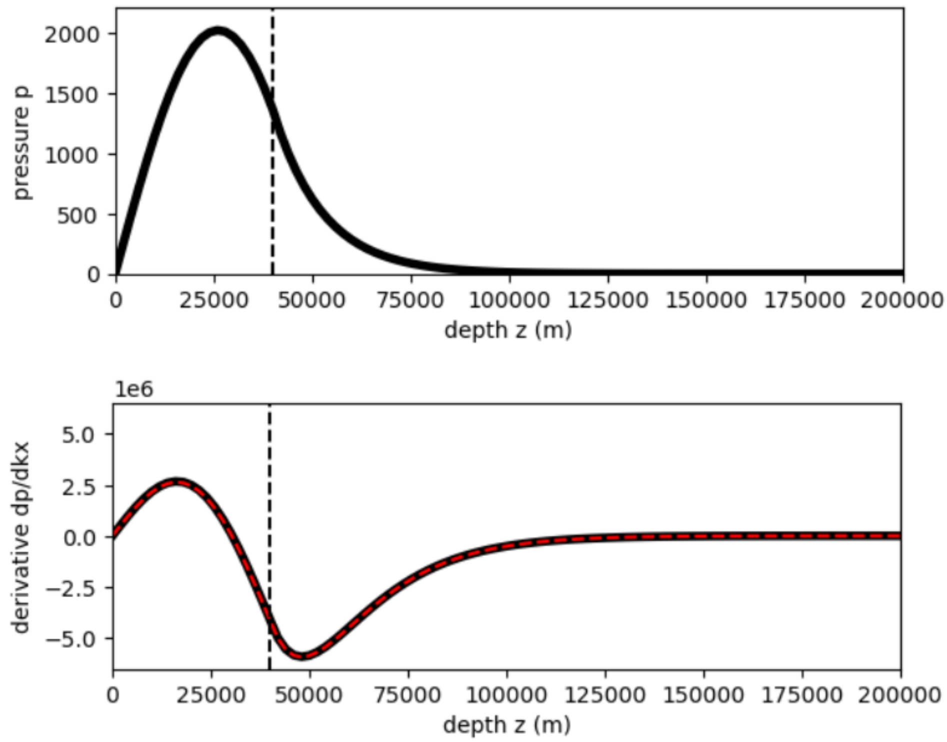
which implies

$$\alpha = -\langle g^{par}(z, k_x^{ref})p(z, k_x^{ref})\rangle$$

(20)

We test this procedure for a layer with $H = 40{,}000$ m, $c_1 = 6{,}500$ m/s and $\rho_1 = 2{,}500$ kg/m^3 above a half-space of $c_2 = 8{,}000$ m/s and $\rho_1 = 3{,}000$ kg/m^3. After specifying a reference frequency $\omega^{ref} =$, the corresponding k_x^{ref} is determined by solving $D(k_x, \omega) = 0$ using Newton's method and the normalized vertical wave-function is constructed (Fig. 1, top). The group velocity $d\omega/dk_x$ is also determined by standard means. The wave-function is then entered into the source term in Eqn. (4), the equation is discretizing using a distance increment $\Delta z = 2000$ m, and solved by standard linear-algebraic means. Finally, the parameter α is calculated by approximating the integral in Eqn. (9) by Riemann's rule, allowing the derivative dp/dk_x (Fig. 1, bottom, black curve) to be constructed via Eqn. (5).

The derivative is tested against the finite difference approximation (Fig. 1, bottom, red-dashed curve), by differencing $p(z, k_x^{ref})$ and $p(z, k_x^{ref} + \Delta k_x)$, where Δk_x is a small increment. The two results agree very well.



(Top) Normalized pressure p as a function of depth z for the fundamental acoustic wave mode at angular frequency $\omega = 1.22\ s^{-1}$. (Bottom) The derivative dp/dk_x, calculated by the direct method (black) and by finite differences (red-dashed curve). Note the excellent agreement between the two methods.

Part 3, Operator and right hand side for the general form of the wave equation

The elastic wave equation is

$$-\omega^2 \rho u_i = \left(c_{ijpq} u_{p,q}\right)_{,j} = c_{ijpq,j} u_{p,q} + c_{ijpq} u_{p,qj} \tag{21}$$

Considering c_{ijpq} to be a function of z, only:

$$-\omega^2 \rho u_i = c_{i3pq,3} u_{p,q} + c_{ijpq} u_{p,qj} \tag{22}$$

Assuming that u_i has no y-dependence

$$-\omega^2 \rho u_i = \left(c_{i3p1,3} u_{p,1} + c_{i3p3,3} u_{p,3}\right) + \left(c_{i1p1} u_{p,11} + c_{i1p3} u_{p,13} + c_{i3p1} u_{p,31} + c_{i3p3} u_{p,33}\right) \tag{23}$$

Assuming a wave-function of the form $u_i = U_i(z)\exp(ik_x x)$

$$-\omega^2 \rho U_i = \left(ik_x c_{i3p1,3} U_p + c_{i3p3,3} U_{p,3}\right) + \left(-k_x^2 c_{i1p1} U_p + ik_x c_{i1p3} U_{p,3} + ik_x c_{i3p1} U_{p,3} + c_{i3p3} U_{p,33}\right) \tag{24}$$

Which simplified to

$$0 = \left(k_x^2 c_{i1p1} U_p - \omega^2 \rho \delta_{ip} - ik_x c_{i3p1,3}\right) U_p - ik_x\left(c_{i3p3,3} + c_{i1p3} + c_{i3p1}\right) U_{p,3} - c_{i3p3} U_{p,33} U_{p,33} \tag{25}$$

In elastic problems, one typically uses the normalization

$$I = \langle U_i^* U_i \rangle \equiv \int_0^\infty \rho U_i^* U_i dz = 1 \tag{26}$$

Taking the derivative $g_p \equiv dU_p/dk_x$ with respect to wavenumber k_x leads to $\mathcal{L}_{ip}\, g_p = f_i$ with $\mathcal{L}_{ip}$ unchanged and

$$f_i = 2\left(k_x c_{i1p1} - \omega \frac{d\omega}{dk_x} \rho \delta_{ip}\right) U_p + i\left(c_{i3p3,3} + c_{i1p3} + c_{i3p1} U_{p,3}\right) U_{p,3} \tag{27}$$

References

Menke, W., 2024, Menke Research Note 284, Frequency Dependence of Rayleigh Wave Amplification by Variation in Earth Structure, https://www.ldeo.columbia.edu/~menke/www_users_menke/research_notes/menke_research_note284.pdf

Menke, W. and D. Abbott, Geophysical Theory (Textbook), Columbia University Press, 458p, 1989