Derivation of the Fréchet Derivative for the Martinson et al. (1982) signal correlation method
Bill Menke, December 4, 2024

Following Martinson et al. (1982), signal correlation is represented by theory

$$d^{(2)}(t) = d^{(1)}\big(T(t,m)\big)$$

It deforms time series $d^{(1)}(t)$ into time series $d^{(2)}(t)$ via a mapping function $T(t,m)$ where m is a model parameter. It is an explicit, nonlinear inverse problem of the form $d(t_i) = g(t_i, m)$.

Our objective is to calculate the derivative

$$\frac{\partial d^{(1)}}{\partial m}$$

so that the problem can be solved using Newton's method.

Exemplary $d^{(1)}(t)$ (blue) and $d^{(2)}(t)$ (red) time series are shown in Fig. 1A. The exemplary mapping function $T(t,m)$ in Fig. 1B has the property that $T(t,m) > t$ when $m > 0$; that is, $d^{(1)}$ is shifted to the right (Fig. 1C) for positive m.

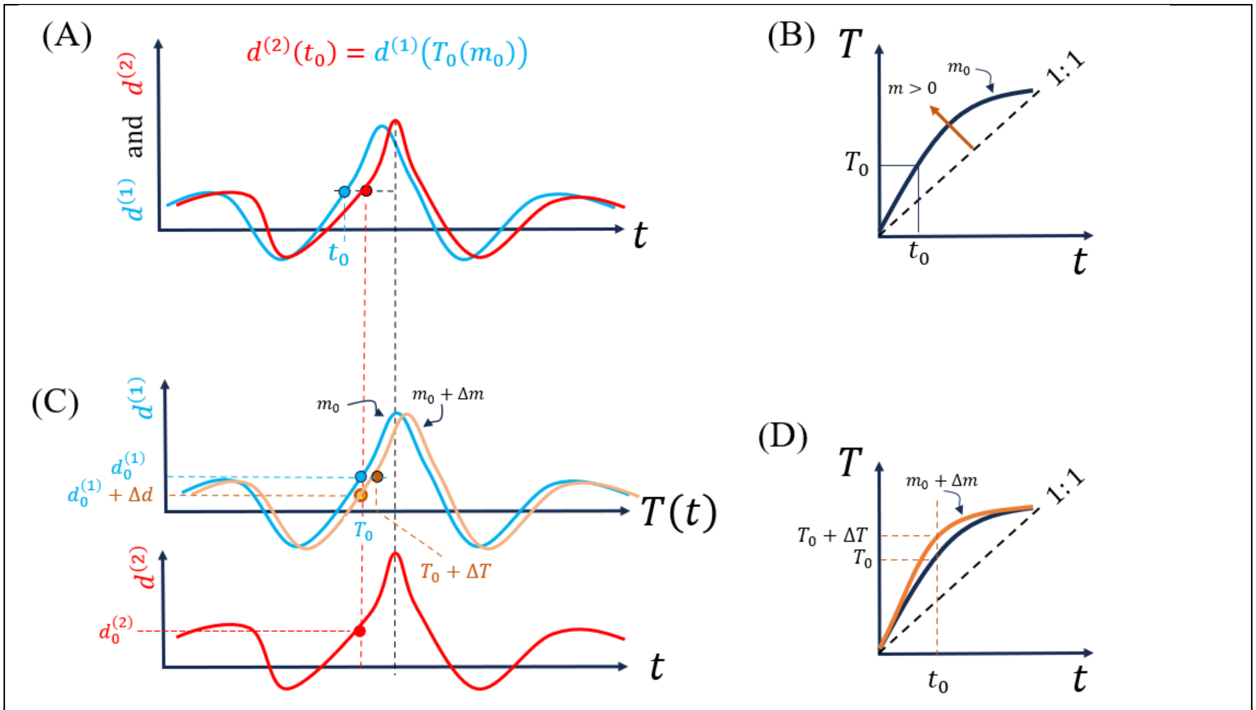

Fig. 1. Diagram used in derivation. See text for further discussion.

Now suppose that the choice $m = m_0$ lines the times series up pretty well in the vicinity of a time t_0. A small perturbation Δm around m_0 causes a small perturbation ΔT in $T_0 \equiv T(t_0, m_0)$ (Fig. 1B).

$$T(t_0, m_0 + \Delta m) \approx T_0 + \frac{\partial T}{\partial m}\Delta m \equiv T_0 + \Delta T$$

The time series $d^{(1)}$ shifts to the right by an amount ΔT; that is, $d^{(1)}(T_0, m_0) = d^{(1)}(T_0 + \Delta T, m_0 + \Delta m)$. At the point T_0 the time series changes by an amount

$$\Delta d^{(1)} = \frac{\partial d^{(1)}}{\partial T}(-\Delta T) = -\frac{\partial d^{(1)}}{\partial T}\frac{\partial T}{\partial m}\Delta m = \frac{\partial d^{(1)}}{\partial m}\Delta m$$

Hence

$$\frac{\partial d^{(1)}}{\partial m} = -\frac{\partial d^{(1)}}{\partial T}\frac{\partial T}{\partial m}$$

Reference

Martinson, D. G., W. Menke, and P. Stoffa, 1982. An inverse approach to signal correlation, J. Geophys. Res. 87, 4807-4818.