**Error in Phase Slowness Estimated by the Two-Station Method**

William Menke, July 25, 2025

The two-station method estimates phase slowness $s$ by using Fourier analysis to determine the phase difference $(\varphi_2 - \varphi_1)$ between observations at two locations, 1 and 2, separated by a distance $X$, and converting it into a slowness using $s = \omega^{-1} X^{-1} (\varphi_2 - \varphi_1)$, where $\omega$ is frequency. Here, I use classical perturbation analysis to estimate the error in such an estimate.

Consider a real field $u(t)$, where $t$ is time, with Fourier transform $\tilde{u}(\omega) \equiv \tilde{u}_R(\omega) + i\tilde{u}_I(\omega)$, where $\omega$ is angular frequency. Its power spectral density (psd) $|\tilde{u}|^2$ and phase $\varphi$ are

$$|\tilde{u}|^2 = \tilde{u}_R^2 + \tilde{u}_I^2 \quad \text{and} \quad \varphi = \tan^{-1}(x) \quad \text{with} \quad z \equiv \frac{-\tilde{u}_I}{\tilde{u}_R} \tag{1}$$

Here, the minus sign accounts for Numpy's sign convention of $+i$ for the inverse Fourier transform.

The first step is to use Taylor's theorem to analyze how a small perturbation $\Delta\tilde{u}$ in $\tilde{u}$ around a reference level $\tilde{u}^{(0)}$ causes a small perturbation $\Delta\varphi$ in phase $\varphi$ around a reference level $\varphi^{(0)}$

$$\varphi = \varphi^{(0)} + \left.\frac{\partial\varphi}{\partial u_R}\right|_0 \left(\tilde{u}_R - \tilde{u}_R^{(0)}\right) + \left.\frac{\partial\varphi}{\partial u_I}\right|_0 \left(\tilde{u}_I - \tilde{u}_I^{(0)}\right) \quad \text{or} \quad \Delta\varphi = \left.\frac{\partial\varphi}{\partial u_R}\right|_0 \Delta\tilde{u}_R + \left.\frac{\partial\varphi}{\partial u_I}\right|_0 \Delta\tilde{u}_I$$

$$\text{with} \quad \Delta\varphi \equiv \varphi - \varphi^{(0)} \quad \text{and} \quad \Delta\tilde{u}_R \equiv \tilde{u}_R - \tilde{u}_R^{(0)} \quad \text{and} \quad \Delta\tilde{u}_I \equiv \tilde{u}_I - \tilde{u}_I^{(0)} \tag{2}$$

The derivatives are:

$$\frac{\partial z}{\partial \tilde{u}_R} = \tilde{u}_I\, \tilde{u}_R^{-2} \quad \text{and} \quad \frac{\partial z}{\partial u_I} = -\tilde{u}_R^{-1} \quad \text{and} \quad \frac{d\varphi}{dz} = \frac{1}{1+z^2}$$

$$\frac{\partial\varphi}{\partial\tilde{u}_R} = \frac{d\varphi}{dz}\frac{\partial z}{d\partial} = \frac{\tilde{u}_I\, \tilde{u}_R^{-2}}{1 + \tilde{u}_I^2 \tilde{u}_R^{-2}} = \frac{\tilde{u}_I}{\tilde{u}_R^2 + \tilde{u}_I^2} = \frac{\tilde{u}_I}{|\tilde{u}|^2}$$

$$\frac{\partial\varphi}{\partial\tilde{u}_I} = \frac{d\varphi}{dz}\frac{dz}{d\tilde{u}_I} = \frac{-\tilde{u}_R^{-1}}{1 + \tilde{u}_I^2 \tilde{u}_R^{-2}} = \frac{-\tilde{u}_R^{-1}\tilde{u}_R^2\tilde{u}_R^{-2}}{1 + \tilde{u}_I^2 \tilde{u}_R^{-2}} = \frac{-\tilde{u}_R}{\tilde{u}_R^2 + \tilde{u}_I^2} = \frac{-\tilde{u}_R}{|\tilde{u}|^2} \tag{3}$$

Hence,

$$\Delta\varphi = \frac{\tilde{u}_I^{(0)}}{\left|\tilde{u}^{(0)}\right|^2}\Delta\tilde{u}_R - \frac{\tilde{u}_R^{(0)}}{\left|\tilde{u}^{(0)}\right|^2}\Delta\tilde{u}_I \tag{4}$$

The slowness $s$ of a plane wave propagating between two locations, $x_1$ and $x_2$ separated by a distance $X \equiv x_2 - x_1$ is

$$s \equiv s_0 + \Delta s \quad \text{with} \quad s_0 = \omega^{-1} X^{-1} \left(\varphi_2^{(0)} - \varphi_1^{(0)}\right) \quad \text{and} \quad \Delta s = \omega^{-1} X^{-1} (\Delta\varphi_2 - \Delta\varphi_1)$$

$$(5)$$

Inserting Eqn. (4) into the formula for $\Delta s$ in Eqn. (5) yields

$$\Delta s = \omega^{-1} X^{-1} \left( -\frac{\tilde{u}_{1I}^{(0)}}{\left|\tilde{u}_1^{(0)}\right|^2} \Delta\tilde{u}_{1R} + \frac{\tilde{u}_{1R}^{(0)}}{\left|\tilde{u}_1^{(0)}\right|^2} \Delta\tilde{u}_{1I} + \frac{\tilde{u}_{2I}^{(0)}}{\left|\tilde{u}_2^{(0)}\right|^2} \Delta\tilde{u}_{2R} - \frac{\tilde{u}_{2R}^{(0)}}{\left|\tilde{u}_2^{(0)}\right|^2} \Delta\tilde{u}_{2I} \right) \tag{6}$$

Now suppose that $\tilde{u}$ is the sum of a deterministic signal $\tilde{u}^{(0)}$ and stationary random noise $\Delta\tilde{u}$. By the usual rule for error propagation, the variance of the phase perturbation is:

$$\operatorname{var} \Delta s = \omega^{-2} X^{-2} (A + B)$$

$$A \equiv \frac{\left[\tilde{u}_{1I}^{(0)}\right]^2}{\left|\tilde{u}_1^{(0)}\right|^4} \operatorname{var} \Delta\tilde{u}_{1R} + \frac{\left[\tilde{u}_{1R}^{(0)}\right]^2}{\left|\tilde{u}_1^{(0)}\right|^4} \operatorname{var} \Delta\tilde{u}_{1I} + \frac{\left[\tilde{u}_{2I}^{(0)}\right]^2}{\left|\tilde{u}_2^{(0)}\right|^4} \operatorname{var} \Delta\tilde{u}_{2R} + \frac{\left[\tilde{u}_{2R}^{(0)}\right]^2}{\left|\tilde{u}_2^{(0)}\right|^4} \operatorname{var} \Delta\tilde{u}_{2I}$$

$$B \equiv -2 \frac{\tilde{u}_{1R}^{(0)} \tilde{u}_{1I}^{(0)}}{\left|\tilde{u}_1^{(0)}\right|^4} \operatorname{cov}(\Delta\tilde{u}_{1R}, \Delta\tilde{u}_{1I}) - 2 \frac{\tilde{u}_{2R}^{(0)} \tilde{u}_{2I}^{(0)}}{\left|\tilde{u}_2^{(0)}\right|^4} \operatorname{cov}(\Delta\tilde{u}_{2R}, \Delta\tilde{u}_{2I})$$

$$-2 \frac{\tilde{u}_{1I}^{(0)} \tilde{u}_{2I}^{(0)}}{\left|\tilde{u}_2^{(0)}\right|^2 \left|\tilde{u}_1^{(0)}\right|^2} \operatorname{cov}(\Delta\tilde{u}_{1R}, \Delta\tilde{u}_{2R}) - 2 \frac{\tilde{u}_{1R}^{(0)} \tilde{u}_{2R}^{(0)}}{\left|\tilde{u}_2^{(0)}\right|^2 \left|\tilde{u}_1^{(0)}\right|^2} \operatorname{cov}(\Delta\tilde{u}_{1I}, \Delta\tilde{u}_{2I})$$

$$\left. +2 \frac{\tilde{u}_{1R}^{(0)} \tilde{u}_{2I}^{(0)}}{\left|\tilde{u}_1^{(0)}\right|^2 \left|\tilde{u}_2^{(0)}\right|^2} \operatorname{cov}(\Delta\tilde{u}_{1I}, \Delta\tilde{u}_{2R}) + 2 \frac{\tilde{u}_{1I}^{(0)} \tilde{u}_{2R}^{(0)}}{\left|\tilde{u}_1^{(0)}\right|^2 \left|\tilde{u}_2^{(0)}\right|^2} \operatorname{cov}(\Delta\tilde{u}_{1R}, \Delta\tilde{u}_{2I}) \right) \tag{7}$$

We assume that the real and imaginary parts of $\Delta\tilde{u}_1$ are uncorrelated and with the equal variance, and similarly for the real and imaginary parts of $\Delta\tilde{u}_2$ (that is, $\Delta\tilde{u}_1$ and $\Delta\tilde{u}_2$ are circular random numbers):

$$\operatorname{var} \Delta\tilde{u}_{1R} = \operatorname{var} \Delta\tilde{u}_{1I} \quad \text{and} \quad \operatorname{var} \Delta\tilde{u}_{2R} = \operatorname{var} \Delta\tilde{u}_{2I} \quad \text{and} \quad \operatorname{cov}(\Delta\tilde{u}_{1R}, \Delta\tilde{u}_{1I}) = \operatorname{cov}(\Delta\tilde{u}_{2R}, \Delta\tilde{u}_{2I}) = 0$$

$$\left|\tilde{u}_{1R}^{(0)}\right|^2 + \left|\tilde{u}_{1I}^{(0)}\right|^2 = \left|\tilde{u}_1^{(0)}\right|^2 \quad \text{and} \quad \left|\tilde{u}_{2R}^{(0)}\right|^2 + \left|\tilde{u}_{2I}^{(0)}\right|^2 = \left|\tilde{u}_2^{(0)}\right|^2 \tag{8}$$

As is well-known, the sum of the variances of the real and imaginary parts of a complex number equals the variance of the complex number, itself:

$$\operatorname{var} \Delta\tilde{u}_{1R} + \operatorname{var} \Delta\tilde{u}_{1I} = \operatorname{var} \Delta\tilde{u}_1 \quad \text{and} \quad \operatorname{var} \Delta\tilde{u}_{2R} + \operatorname{var} \Delta\tilde{u}_{2I} = \operatorname{var} \Delta\tilde{u}_2 \tag{9}$$

Because the perturbation is stationary, its variance is independent of location. In principle, the power $\left|\tilde{u}^{(0)}\right|^2$ of the unperturbed field can vary spatially. However, we assume here that the separation distance $X$ is sufficiently small that the power is approximately constant:

$$\operatorname{var}\Delta\tilde{u}_{1R} = \operatorname{var}\Delta\tilde{u}_{2R} \quad \text{and} \quad \operatorname{var}\Delta\tilde{u}_{1R} = \operatorname{var}\Delta\tilde{u}_{2I}$$

$$\operatorname{var}\Delta\tilde{u}_{1R} = \operatorname{var}\Delta\tilde{u}_{2R} = \operatorname{var}\Delta\tilde{u}_{1R} = \operatorname{var}\Delta\tilde{u}_{2I} = ½\operatorname{var}\Delta\tilde{u}$$

$$\left|\tilde{u}_1^{(0)}\right|^2 = \left|\tilde{u}_2^{(0)}\right|^2 = \left|\tilde{u}^{(0)}\right|^2$$

(10)

With these assumptions, the factor $A$ simplifies to:

$$A = \frac{\operatorname{var}\Delta\tilde{u}}{\left|\tilde{u}^{(0)}\right|^2} = \frac{1}{R^2} \quad \text{with} \quad R \equiv \frac{\left|\tilde{u}^{(0)}\right|}{\sqrt{\operatorname{var}\Delta\tilde{u}}}$$

(11)

Here, $R$ is the signal-to-noise ratio.

In the simplest case in which the noise $\Delta\tilde{u}$ is uncorrelated between the two locations, $B = 0$ and $\operatorname{var}\Delta s = \omega^{-2}X^{-2}R^{-2}$. In the limit $X \to \infty$, $\operatorname{var}\Delta s \to \infty$.

Now we examine the case where the perturbation $\Delta\tilde{u}$ has spatial correlation. Because $\Delta\tilde{u}_R$ and $\Delta\tilde{u}_I$ are assumed uncorrelated when both are measured at a single location $x_1$, one would not expect correlation to be introduced by measuring them at two different points, $x_1$ and $x_2$. Hence

$$\operatorname{cov}(\Delta\tilde{u}_{1R}, \Delta\tilde{u}_{1I}) = \operatorname{cov}(\Delta\tilde{u}_{2R}, \Delta\tilde{u}_{2I}) = 0$$

(12)

The real and imaginary parts of $\Delta\tilde{u}$ play completely symmetric roles, so we would expect their covariance to be equal.

$$\operatorname{cov}(\Delta\tilde{u}_{1R}, \Delta\tilde{u}_{2R}) = \operatorname{cov}(\Delta\tilde{u}_{1I}, \Delta\tilde{u}_{2I})$$

(13)

With these assumptions

$$C = -2\frac{\left(\tilde{u}_{1R}^{(0)}\tilde{u}_{2R}^{(0)} + \tilde{u}_{1I}^{(0)}\tilde{u}_{2I}^{(0)}\right)}{\left|\tilde{u}^{(0)}\right|^4}\operatorname{cov}(\Delta\tilde{u}_{1R}, \Delta\tilde{u}_{2R})$$

(14)

Suppose that $u^{(0)}$ is a plane wave with a Fourier transform of the form

$$\tilde{u}_{1R}^{(0)} = \left|\tilde{u}^{(0)}\right|\cos(kx+\theta) \quad \text{and} \quad \tilde{u}_{1I}^{(0)} = \left|\tilde{u}^{(0)}\right|\sin(kx+\theta)$$

$$\tilde{u}_{2R}^{(0)} = \left|\tilde{u}^{(0)}\right|\cos(kx+\theta+kX) \quad \text{and} \quad \tilde{u}_{1I}^{(0)} = \left|\tilde{u}^{(0)}\right|\sin(kx+\theta+kX)$$

(15)

Here, $k = \omega s_0$ is wavenumber and $\theta$ is an overall phase. The trigonometric identity $\cos(b-a) = \cos(a)\cos(a) + \sin(a)\sin(b)$ implies

$$\tilde{u}_{1I}^{(0)}\tilde{u}_{2I}^{(0)} + \tilde{u}_{1R}^{(0)}\tilde{u}_{2R}^{(0)} = \left|\tilde{u}^{(0)}\right|^2 \cos(kX)$$

(16)

and we find

$$B = -2\frac{\cos(kX)}{\left|\tilde{u}^{(0)}\right|^2}\operatorname{cov}(\Delta\tilde{u}_{1R}, \Delta\tilde{u}_{2R})$$

(17)

Then

$$\operatorname{var}\Delta s = \omega^{-2}X^{-2}R^{-2}(1 - C\cos(kX)) \quad \text{with} \quad C = \frac{\operatorname{cov}(\Delta\tilde{u}_{1R}, \Delta\tilde{u}_{2R})}{\operatorname{var}\Delta\tilde{u}_R}$$

(18)

Here, $C$ is a spatial correlation coefficient. Aki (1957, eqn. 42) showed that for micro-seismic noise dominated by surface waves with wavenumber $k$

$$C = J_0(kX)$$

(19)

Here, $J_0$ is the zeroth-order Bessel function of the first kind. We conclude that the relative error in phase velocity is:

$$\frac{\sigma_s}{s_0} \equiv \frac{\sqrt{\operatorname{var}\Delta s}}{s_0} = (kX)^{-1}R^{-1}\sqrt{F} \text{ with } F \equiv (1 - J_0(kX)\cos(kX))$$

(20)

Here, we have used $k\omega^{-1} = s_0$. Note that $\max_{kX} J_0(kX) = \max_{kX} J_0(kX) = 1$, implying that $F$ is never negative. In the limit $(kX) \to 0$, $\cos(kX) \approx 1 - ½(kX)^2$ and $J_0(kX) \approx 1 - ¼(kX)^2$ so $F \approx ¾(kX)^2$ . In this limit, the relative error is:

$$\frac{\sigma_s}{s_0} = \frac{\sqrt{3}}{2}\frac{1}{R} \approx \frac{0.866}{R}$$

The relative error in slowness is proportional to the reciprocal of the signal-to-noise ratio. In the limit $(kX) \to \infty$, $J_0(kX) \to 0$ and $(\operatorname{var}\Delta s)/s_0^2 \to 0$. However, this latter results overlooks the problem of unwrapping the phase when the two locations are many wavelengths apart. An exemplary plot of $\sigma_s/s_0$ is shown in Figure 1.

The coherence structure of the noise is shown to be extremely important in determining the relative error in slowness for small separation distances (that is, $(kX) \ll 1$). Uncorrelated (e.g. electronic noise in the seismometer), leads to indefinitely large error at small separation, because noise-induced phase shifts overwhelm the small phase differences between the two signal. In contrast, when the noise is spatially correlated, the noise-induced phase shifts cancel at small offset. When the cancellation is strong enough, the relative error can reach a finite limit. If, when $(kX) \ll 1$, $C \approx 1 - \gamma(kX)^n$, where $\gamma$ is a positive constant and $n$ is a positive integer, then the relative error reaches a finite limit only when $n \geq 2$ . Thus, $C = J_0(kX)$ leads to a finite error, whereas $C = \exp(-kX)$ does not.

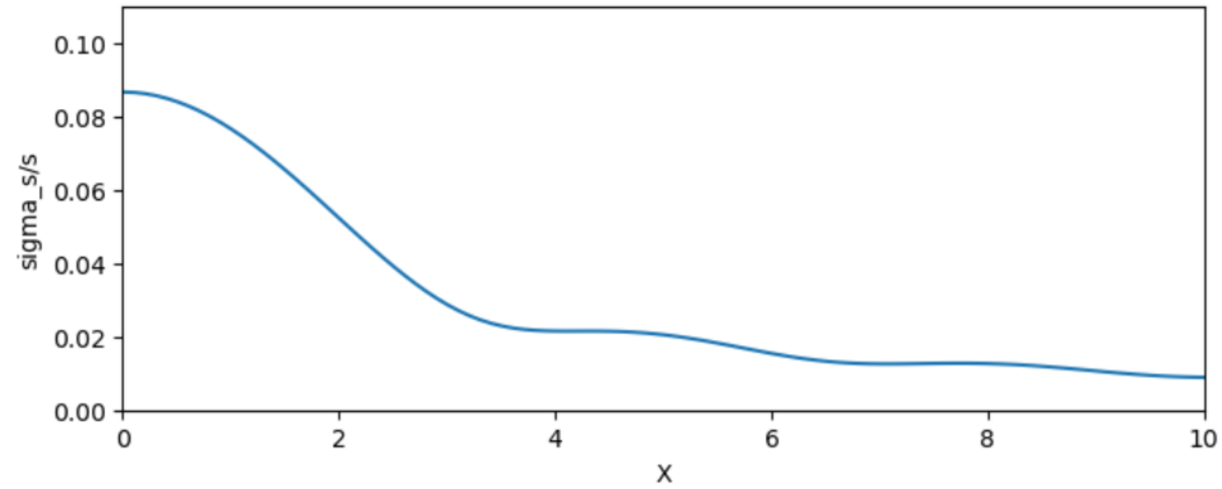


Fig. 1. Relative error in slowness $\sigma_s/s_0$ as a function of separation distance $X$ for $k = 1$ for a signal-to-noise ratio of $R = 10$.

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