Estimated Phase Velocity for a Very Simple Rayleigh Wave Scattering Model
by Bill Menke after discussion with Eva Cullen
July 25, 2025

Suppose a harmonic plane Rayleigh wave of amplitude U is propagating in the positive x-direction (Fig. 1A). It interacts with a very small heterogeneity at $x = 0$. The scattering interaction is modeled by assuming back and forward scattered waves of equal amplitude $UR|x + \varepsilon^2|^{-½}$ (where ε is a small number) and a phase such that the forward scattered wave appears to have originated from $x - x_0$ and the back scattered wave from $x + x_0$. The wave field at position x_i has the form:

$$u_i = \begin{cases} U\left[\exp(ikx_i) - R|x_i + \varepsilon^2|^{-½} \exp\left(+ik(x_i - x_0)\right)\right] \exp(-i\omega t) & \text{when } x_i \geq 0 \\ U\left[\exp(ikx_i) - R|x_i + \varepsilon^2|^{-½} \exp\left(-ik(x_i + x_0)\right)\right] \exp(-i\omega t) & \text{when } x_i < 0 \end{cases}$$

(1)

Here, U is the amplitude of the incident wave, R is the scattering coefficient, k is wavenumber and ω is angular frequency. Note that the distance x_0 is equivalent to a time delay of $\Delta t = kx_0/\omega = x_0/c$, as $c = \omega/k$. A low velocity anomaly is associated with $R > 0$ and $\Delta t > 0$.

For two neighboring stations, x_j and x_i, separated by distance $\Delta x \equiv x_j - x_i$, the differential phase is

$$\varphi = \tan^{-1}\left(\frac{\operatorname{imag} r}{\operatorname{real} r}\right) \text{ with } r \equiv +\frac{u_j}{u_i}$$

(2)

Here the $+$ sign is appropriate for a inverse Fourier transform containing $-i$. The “apparent” wavenumber estimated from this phase is $k^{est} = \varphi/\Delta x$ (where $\Delta x \equiv x_j - x_i$) and the corresponding estimated phase velocity is $c^{est} = \omega/k^{est}$. In general, $c^{est} \neq c$; that is, the scattering interaction perturbs the apparent phase velocity.

We performed a numerical experiment (Fig. 1B,C) in which we computed the a wave field for the interval $-10 < x < 10$, for a slow velocity anomaly modeled by $R = -0.01$ and $\Delta t = 0.01$ and $\Delta x = 0.1$.

In the low frequency ($\omega = 0.1$) case, the estimated velocity for xs just to the right of the origin is fast, in contrast to the expected slower velocity. Velocity for xs just to the left of the origin are slow.

The anomalous behavior at $x > 0$ is not present in the high frequency ($\omega = 0.5$) case; the estimated velocity is very nearly c. Velocity for xs just to the left of the origin are slow but at greater negative distances are oscillatory.

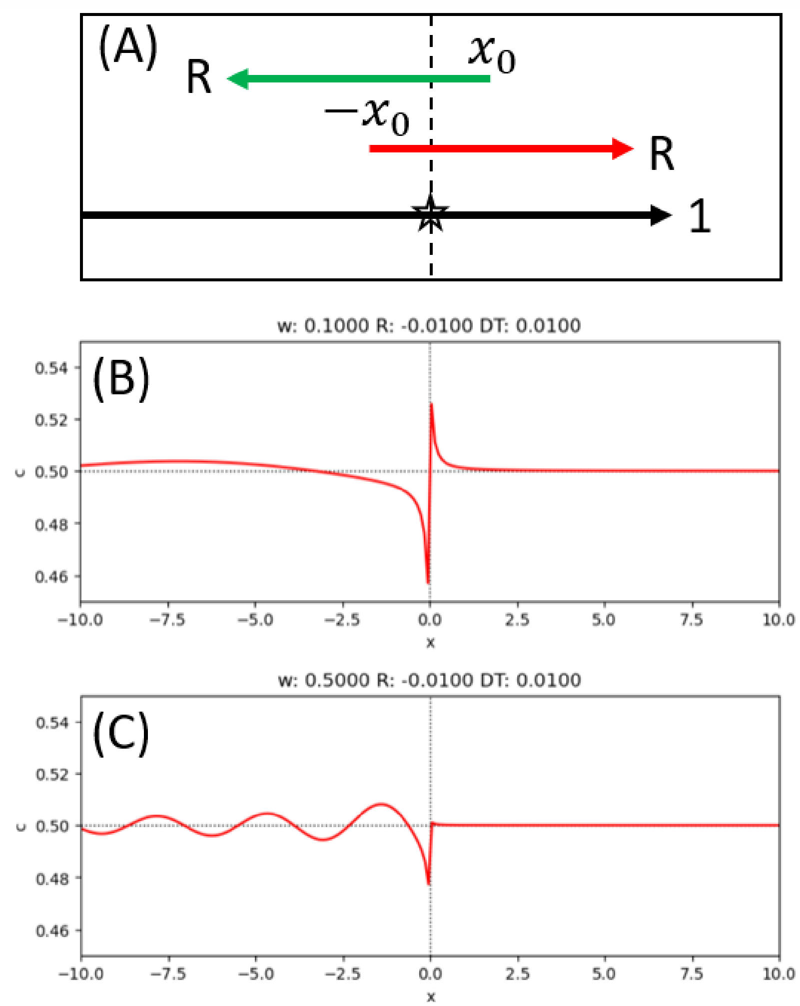

Fig. 1. (A) Simple scattering model. Incident wave (black) interacts with a point scatterer at $x = 0$ (star), generating back scattered (green) and forward scattered waves of the same amplitude, R. Delays in the back and forward scattered waves are modeled by introducing a delay ΔT equivalent to them having originated at positions $-x_0$ and $+x_0$, respectively. (B) Estimated phase velocity at low frequency. (C) Estimated phase velocity at high frequency.