Addendum to Research Note 293
The Differential Phase Method for Estimating Surface Wave Phase Velocity

Bill Menke, September 19, 2025

Two ideas that improve the method:

(1) A method to improve the phase-unwrapping step, when one has a rough idea of the differential travel time between the vertices of the triangular array of stations (that is, one roughly knowns the phase velocity and direction of propagation).. The idea is to time shift one of member of a pair of seismograms by an amount that approximately accounts for the propagation time between them before computing their phase difference. This process reduces the number of phase-jumps encountered during the un-wrapping process. The resulting phase difference is then corrected for the effect of the time shift.

Suppose that one has a rough idea of the direction of propagation θ^{est} and phase velocity v^{est}. A unit vector parallel to the direction of propagation is $\mathbf{t}^{est} = [\sin\theta^{est}, \cos\theta^{est}]^T$. The triangular array is defined by three stations (i, j, k) at positions $\mathbf{x}^{(i)}$, $\mathbf{x}^{(j)}$ and $\mathbf{x}^{(j)}$. Now consideration stations (i, j) with separation vector $\Delta\mathbf{x}^{(ji)} \equiv \mathbf{x}^{(j)} - \mathbf{x}^{(i)}$. The differential travel time between these two stations is approximately

$$\Delta t_{ji}^{est} = \frac{\Delta\mathbf{x}^{(ji)} \cdot \mathbf{t}^{est}}{v^{est}}$$

Now suppose that the seismograms $u^{(i)}(t)$, where t is time, have Fourier transform $\tilde{u}^{(i)}(\omega)$, where ω is angular frequency. In the usual practice, the phase difference is calculated as

$$\Delta\varphi^{(ji)} = \mathrm{U}(\Phi(\xi)) \quad \text{with} \quad \xi \equiv \frac{\tilde{u}^{(j)}}{\tilde{u}^{(i)}}$$

Here, $\Phi(x)$ is the phase of the complex number x, defined as

$$\Phi(x) \equiv \mathrm{arctan2}(-\,\mathrm{imag}\, x, \mathrm{real}\, x)$$

and $\mathrm{U}(.)$ denotes the unwrapping version of phase $\Phi(.)$. The unwrapping is necessary because $\Phi\big(\xi(\omega)\big)$ contains $\pm 2\pi$ phase jumps due to $\big(\Delta\mathbf{x}^{(ji)} \cdot \mathbf{t}^{est}\big)$ spanning more than one wavelength. The unwrapping process can be facilitated by time-shifting $u^{(j)}$ by an amount $-t_{ji}^{est}$, which is to say, introducing the phase factor $\exp\big(i\omega\Delta t_{ji}^{est}\big)$. In effect, the position of station j is shifted to the position of station i before the phase calculation, with the shift being undone, afterwards:

$$\Delta\varphi_0^{(ji)} \equiv \mathrm{U}(\Phi(\xi')) + \omega\Delta t_{ji}^{est} \quad \text{with} \quad \xi' \equiv \frac{\tilde{u}^{(j)} \exp\big(i\omega\Delta t_{ji}^{est}\big)}{\tilde{u}^{(i)}}$$

Although the modified phase $\Phi(\xi')$ still needs to be unwrapped, it will contain many fewer phase jumps than does the unmodified phase. (The signs here are for Python's definition of the Fourier Transform, which contains a $(-i)$ phase factor).

In practice, seismograms are dominated by noise at the longest periods, so unwrapping phase by starting at zero frequency and working towards higher frequencies produces unreliable results. Instead, we identify an angular frequency, say ω_0, that is high enough to be reliable but low enough to contain no phase jumps. We then unwrap $\Phi(\xi'(\omega))$ forward in frequency for $\omega > \omega_0$ and backward in frequency for $\omega < \omega_0$. This process ensures that $U(\Phi(\xi'(\omega_0)))$ is in its original $(-\pi, +\pi)$ interval.

(2) Accounting for ellipticity of the Earth. Consider a triangular array of stations (i, j, k) at (latitudes, longitudes) of $(lat^{(i)}, lon^{(i)})$, $(lat^{(j)}, lon^{(j)})$ and $(lat^{(k)}, lon^{(k)})$, measured in degrees. Let the center of the triangle be defined as

$$lat^{(c)} \equiv \frac{lat^{(i)} + lat^{(j)} + lat^{(k)}}{3} \quad \text{and} \quad lon^{(c)} \equiv \frac{lon^{(i)} + lon^{(j)} + lon^{(k)}}{3}$$

In a spherical Earth, a locally planar transform to Cartesian coordinates, measured in km, is

$$\mathbf{x}^{(i)} \equiv \begin{bmatrix} A\left[lon^{(i)} - lon^{(c)}\right] \\ B\left[lat^{(i)} - lat^{(c)}\right] \end{bmatrix} \quad \text{with} \quad B = 111.12 \quad \text{and} \quad A = B\cos\left(lat^{(c)}\right)$$

and similarly, for j and k. However, this formula has significant error when applied to an elliptical Earth, unless the factors A and B (and especially B) are modified to account for the ellipticity. Estimates of them are obtained using the standard function, arcl2( ), which returns the distance between two points on the Earth's surface:

$$A = \text{arcl2}\left(lat^{(c)}, lon^{(c)} - ½, lat^{(c)}, lon^{(c)} + ½\right)$$

$$B = \text{arcl2}\left(lat^{(c)} - ½, lon^{(c)}, lat^{(c)} + ½, lon^{(c)}\right)$$

A test of the method for synthetic seismograms with a realistic source-receiver geometry is shown in Figures 1-4.

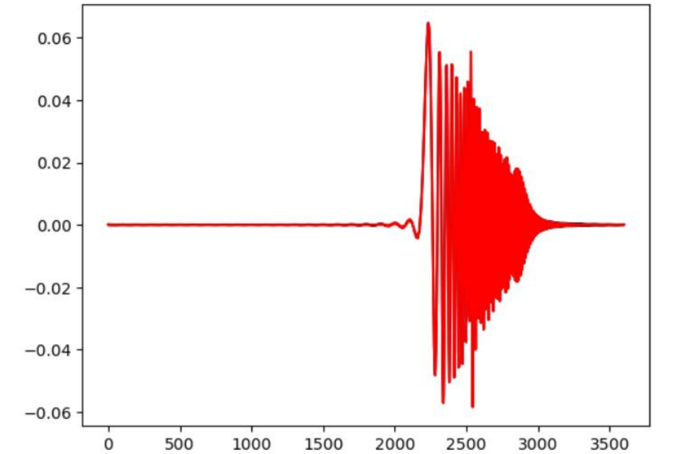

Fig. 1. Synthetic Rayleigh wave seismogram for an event in the Kurile Islands (Russia) observed in New England (USA).

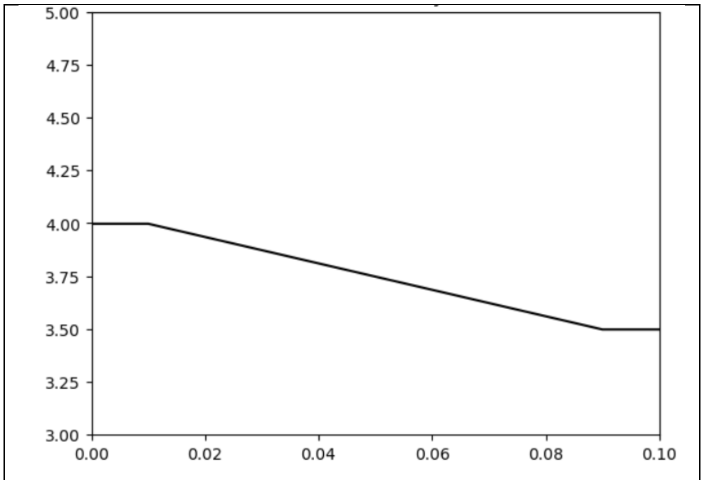

Figure 2. Phase velocity vs. frequency for the synthetic seismogram shown in Fig. 1. The velocity at a frequency of at 0.01 Hz is 4.00 km/s.

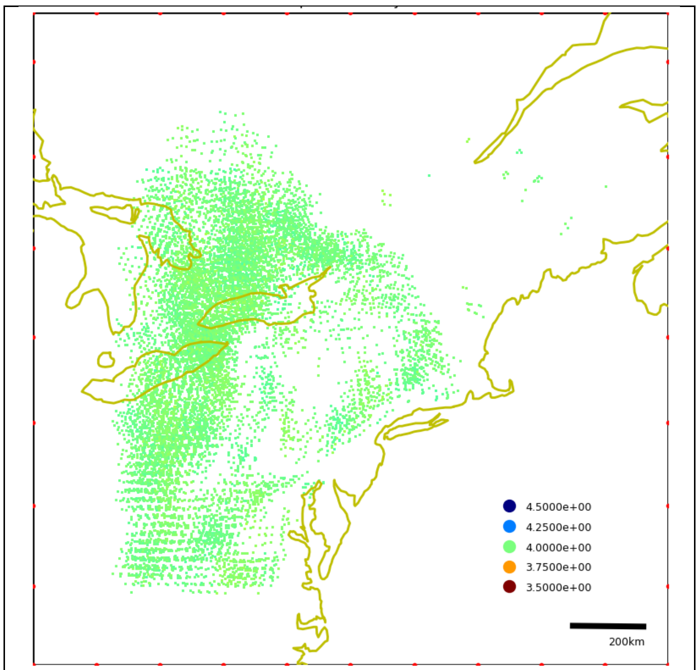

Fig. 3. Estimate phase velocities at a frequency of 0.01 Hz, plotted at the center of each triangular array (dots). Estimates closely match the theoretical phase velocity of 4.00 km/s (green color)

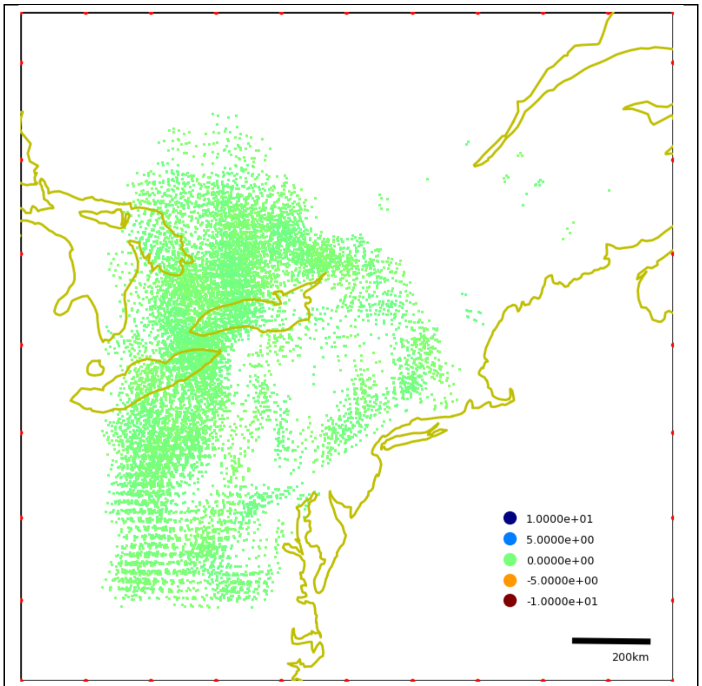

Fig. 3. Direction of propagation anomalies at a frequency of 0.01 Hz, plotted at the center of each triangular array (dots). The anomalies are within a few tenths of a degree of zero (green color).