Adjoint Sources for Time and Space Sensitivities are the Same

Bill Menke June 1, 2026

That “the adjoint sources for time and space sensitivities are the same” is well-known. Here, I use a very compact notation to demonstrate that it is so.

Consider a wave equation of the form:

$$\mathcal{L}u = \left(\rho\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x}\kappa\frac{\partial}{\partial x}\right)u = f$$

where (ρ, κ) are material parameters, together with a waveform error E (the L_2 norm of individual errors $e \equiv u^{obs} - u$). The error sensitivity derivatives are

$$\frac{\delta E}{\delta \rho} \text{ and } \frac{\delta E}{\delta \kappa}$$

I refer to $\delta E/\delta\rho$ as the time-term sensitivity, as it arises from the part of the wave equation containing $\partial/\partial t$ and $\delta E/\delta\kappa$ as the space-term sensitivity, as it arises from the part containing $\partial/\partial x$. Here, I show that although the adjoint-based formulas for these two sensitivities are different, the adjoint sources in those two formulae are the same.

Part 1. Sensitivities for the 1D isotropic wave equation. The 1D isotropic wave equation is:

$$\mathcal{L}u = \left(\rho\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x}\kappa\frac{\partial}{\partial x}\right)u = f$$

The sensitivity derivative for a parameter m_k is:

$$\frac{\partial E}{\partial m_k} = -2\left(e, \frac{\partial u}{\partial m_k}\right)_{x,t}$$

and the derivative of the wavefield with respect to a parameter m_k is:

$$\frac{\partial u}{\partial m_k} = -\mathcal{L}^{-1}\frac{\partial \mathcal{L}}{\partial m_k}u$$

Combining these equations yields:

$$\frac{\partial E}{\partial m_k} = 2\left(e, \mathcal{L}^{-1}\frac{\partial \mathcal{L}}{\partial m_k}u\right)_{x,t} = 2\left(\mathcal{L}^{\dagger-1}e, \mathcal{L}^{-1}\frac{\partial \mathcal{L}}{\partial m_k}u\right)_{x,t} = 2\left(\lambda, \frac{\partial \mathcal{L}}{\partial m_k}u\right)_{x,t}$$

Here, we have defined an adjoint field $\lambda \equiv \mathcal{L}^{\dagger-1}e$ that satisfies the adjoint wave equation $\mathcal{L}^{\dagger}\lambda = e$. Consequently, the adjoint source is e. This expression makes clear that differences between $\partial E/\partial m_k$ for time and space term parameters arise from $\partial\mathcal{L}/\partial m_k$ and not from λ or e.

Case 1: Time-term sensitivity. The time-term material parameter $\rho(x)$ is represented as a background field $\rho_0(x)$ plus a point perturbation $m_k \delta\left(x - x^{(k)}\right)$ located at position $x^{(k)}$:

$$\mathcal{L} = \left(\rho_0 + m_k \delta\left(x - x^{(k)}\right)\right) \frac{\partial^2}{\partial t^2} + \cdots$$

The derivative of the operator is:

$$\frac{\partial \mathcal{L}}{\partial m_k} = \delta\left(x - x^{(k)}\right) \frac{\partial^2}{\partial t^2}$$

Inserting this result into the formula for the error sensitivity derivative yields:

$$\frac{\partial E}{\partial m_k} = 2\left(\lambda, \frac{\partial \mathcal{L}}{\partial m_k} u\right)_{x,t} = 2\left(\lambda, \delta\left(x - x^{(k)}\right) \frac{\partial^2 u}{\partial t^2}\right)_{x,t} = 2\left(\lambda\left(x^{(k)}\right), \frac{\partial^2 u\left(x^{(k)}\right)}{\partial t^2}\right)_t$$

or

$$\frac{\partial E}{\partial m_k} = 2\left(\lambda\left(x^{(k)}\right), \ddot{u}\left(x^{(k)}\right)\right)_t$$

Case 2: Space-term error sensitivity. The space-term material parameter $\kappa(x)$ is represented as a background field $\kappa_0(x)$ plus a point perturbation $m_k \delta\left(x - x^{(k)}\right)$ located at position $x^{(k)}$:

$$\mathcal{L} = \cdots - \frac{\partial}{\partial x}\left(\kappa_0 + m_k \delta\left(x - x^{(k)}\right)\right) \frac{\partial}{\partial x}$$

Its derivative is:

$$\frac{\partial \mathcal{L}}{\partial m_k} = -\frac{\partial}{\partial x} \delta\left(x - x^{(k)}\right) \frac{\partial}{\partial x}$$

And the sensitivity derivative is:

$$\frac{\partial E}{\partial m_k} = 2\left(\lambda, \frac{\partial \mathcal{L}}{\partial m_k} u\right)_{x,t} = -2\left(\lambda, \frac{\partial}{\partial x} \delta\left(x - x^{(k)}\right) \frac{\partial u}{\partial x}\right)_{x,t} = 2\left(\frac{\partial \lambda}{\partial x}, \delta\left(x - x^{(k)}\right) \frac{\partial u}{\partial x}\right)_{x,t}$$

The spatial inner product can be trivially integrated, leaving the temporal inner product (or "correlation");

$$\frac{\partial E}{\partial m_k} = 2\left(\frac{\partial \lambda\left(x^{(k)}\right)}{\partial x}, \frac{\partial u\left(x^{(k)}\right)}{\partial x}\right)_t$$

Note that this formula contains the spatial derivative $\partial\lambda/\partial x$ of the adjoint field, and not the adjoint field, itself.

The derivation for the 3D anisotropic wave equation is similar. The wave equation is:

$$\ddot{\rho}u_i - \left(C_{ijpq}u_{p,q}\right)_{,j} = f_i \quad \text{or} \quad \boldsymbol{\mathcal{L}}\mathbf{u} = \left(\rho\frac{\partial^2}{\partial t^2}\mathbf{I} - \nabla\cdot(\mathbf{C} : \nabla)\right)\mathbf{u} = \mathbf{f}$$

Here, the second order tensor with elements $C_{ijpq}u_{p,q}$ is denoted $\mathbf{C} : \nabla\mathbf{u}$. The sensitivity derivative and operator derivative are:

$$\frac{\partial E}{\partial m_k} = -2\left(\mathbf{e}, \frac{\partial\mathbf{u}}{\partial m_k}\right)_{\mathbf{x},t} \quad \text{and} \quad \frac{\partial\mathbf{u}}{\partial m_k} = -\boldsymbol{\mathcal{L}}^{-1}\frac{\partial\boldsymbol{\mathcal{L}}}{\partial m_k}\mathbf{u}$$

$$\frac{\partial E}{\partial m_k} = 2\left(\boldsymbol{\mathcal{L}}^{\dagger-1}\mathbf{e}, \frac{\partial\boldsymbol{\mathcal{L}}}{\partial m_k}\mathbf{u}\right)_{\mathbf{x},t} = 2\left(\boldsymbol{\lambda}, \frac{\partial\boldsymbol{\mathcal{L}}}{\partial m_k}\mathbf{u}\right)_{\mathbf{x},t}$$

with $\boldsymbol{\mathcal{L}}^{\dagger}\boldsymbol{\lambda} = \mathbf{e}$.

Case 1: Time-term sensitivity. The time-term part of the differential operator is:

$$\boldsymbol{\mathcal{L}} = \left(\rho_0 + m_k\delta\left(\mathbf{x} - \mathbf{x}^{(k)}\right)\right)\frac{\partial^2}{\partial t^2}\mathbf{I} + \cdots$$

and its derivative is

$$\frac{\partial\boldsymbol{\mathcal{L}}}{\partial m_k} = \delta\left(\mathbf{x} - \mathbf{x}^{(k)}\right)\frac{\partial^2}{\partial t^2}\mathbf{I}$$

Consequently, the sensitivity derivative is:

$$\frac{\partial E}{\partial m_k} = 2\left(\boldsymbol{\lambda}, \frac{\partial\boldsymbol{\mathcal{L}}}{\partial m_k}\mathbf{u}\right)_{\mathbf{x},t} = 2\left(\boldsymbol{\lambda}, \delta\left(\mathbf{x} - \mathbf{x}^{(k)}\right)\ddot{\mathbf{u}}\right)_{\mathbf{x},t} = 2\left(\boldsymbol{\lambda}\left(\mathbf{x}^{(k)}\right), \ddot{\mathbf{u}}\left(\mathbf{x}^{(k)}\right)\right)_{\mathbf{x}}$$

Case 2: Space-term sensitivity. The space-term part of the differential operator is:

$$\boldsymbol{\mathcal{L}} = \cdots - \nabla\cdot\left(\left(\mathbf{C_0} + m_{kmnrs}\delta\left(x - x^{(k)}\right)\mathbf{D}^{(mnrs)}\right) : \nabla\right)$$

Here, the model parameter m_{kmnrs} represents a perturbation at position $x^{(k)}$ of the $mnrs$ element of $\mathbf{C}$. The fourth order tensor $\mathbf{D}^{(mnrs)}$ is zero except for its $mnrs$ element, which is unity. The derivative of the operator is:

$$\frac{\partial\boldsymbol{\mathcal{L}}}{\partial m_{kmnrs}} = \cdots - \nabla\cdot\left(\delta\left(x - x^{(k)}\right)\mathbf{D}^{(mnrs)} : \nabla\right)$$

and the sensitivity derivative is:

$$\frac{\partial E}{\partial m_{kmnts}} = -2\left(\boldsymbol{\lambda}, \nabla\cdot\left(\delta\left(x - x^{(k)}\right)\mathbf{D}^{(mnrs)} : \nabla\mathbf{u}\right)\right)_{\mathbf{x},t} = 2\left(\nabla\boldsymbol{\lambda}\left(x^{(k)}\right), \mathbf{D}^{(klmn)} : \nabla\mathbf{u}\left(x^{(k)}\right)\right)_t$$

Here, we have used the rule that the adjoint of $(\nabla\,\cdot)$ is $-\nabla$. Like the 1D case, the 3D case also involves the spatial derivative of the adjoint field.

Data sensitivities. Nothing in the above adjoint manipulations require any special relationship between $\mathbf{e}$ and $\mathbf{u}$. Consequently, if we were to assert that some datum d is related to the wavefield $\mathbf{u}$ by $d = (\mathbf{h}, \mathbf{u})_{x,t}$, where $\mathbf{h}$ is a known vector field, then the data sensitivity derivative would be:

$$\frac{\partial d}{\partial m_k} = -\left(\boldsymbol{\lambda}, \frac{\partial \boldsymbol{\mathcal{L}}}{\partial m_k} \mathbf{u}\right)_{x,t} \quad \text{with} \quad \boldsymbol{\mathcal{L}}^{\dagger} \boldsymbol{\lambda} = \mathbf{h}$$

Hence the results, that "adjoint sources for time and space sensitivities are the same" apply to data derivatives, too.