When an Inner Product Involves a Component, the Adjoint Field is Parallel to the Component

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Summary: When an inner product involves a component of the field, the adjoint field is just the kernel promoted to a vector parallel to the component.

Suppose that $u \equiv \mathbf{t}^T\mathbf{u}$ is a scalar component of a vector wavefield $\mathbf{u} \equiv \left[u_x, u_y, u_z\right]^T$ in the direction $\mathbf{t}$. For example, the vertical component u_z is given by $\mathbf{t} \equiv [0,0,1]^T$ and the radial component u_r by $\mathbf{t} \equiv [\cos(\theta), -\sin(\theta), 1]^T$, where θ is back-azimuth.

Consider an inner product:

$$p \equiv \left(\frac{\partial u}{\partial m}, s\right)_{\mathbf{x},t} \quad \text{with} \quad s \equiv \mathcal{F}u$$

where m is a model parameter, $\mathcal{F}$ is a linear operator and we will refer to s as a kernel. Inner products of this form appear in many inverse theory problems. Typically, m parameterizes a material parameter in the differential equation:

$$\boldsymbol{\mathcal{L}}(m)\,\mathbf{u}(m) = \mathbf{q}$$

where $\boldsymbol{\mathcal{L}}$ is a linear differential operator and $\mathbf{q}$ is a source function. The derivative of the scalar component is given by

$$\frac{\partial u}{\partial m} = \frac{\partial}{\partial m}(\mathbf{t}^T\mathbf{u}) = \mathbf{t}^T\frac{\partial \mathbf{u}}{\partial m}$$

The Born approximation is:

$$\frac{\partial \mathbf{u}}{\partial m} = \frac{\partial}{\partial m}(\boldsymbol{\mathcal{L}}^{-1}\mathbf{q}) = \frac{\partial \boldsymbol{\mathcal{L}}^{-1}}{\partial m}\mathbf{q} = -\boldsymbol{\mathcal{L}}^{-1}\frac{\partial \boldsymbol{\mathcal{L}}^{-1}}{\partial m}\boldsymbol{\mathcal{L}}^{-1}\mathbf{q} = -\boldsymbol{\mathcal{L}}^{-1}\frac{\partial \boldsymbol{\mathcal{L}}^{-1}}{\partial m}\mathbf{u}$$

Suppose that one forms a vector field $\mathbf{s} \equiv \mathcal{F}\mathbf{u}$ where $\mathcal{F}$ is understood to act component-wise. Then, $\mathbf{t}^T\mathbf{s} = \mathbf{t}^T\mathcal{F}\mathbf{u} = \mathcal{F}(\mathbf{t}^T\mathbf{u}) = \mathcal{F}u = s$. Then,

$$p = \left(\frac{\partial u}{\partial m}, s\right)_{\mathbf{x},t} = -\left(\mathbf{t}^T\boldsymbol{\mathcal{L}}^{-1}\frac{\partial \boldsymbol{\mathcal{L}}^{-1}}{\partial m}\mathbf{u}, \mathbf{t}^T\mathbf{s}\right)_{\mathbf{x},t} = -\left(\frac{\partial \boldsymbol{\mathcal{L}}^{-1}}{\partial m}\mathbf{u}, \boldsymbol{\mathcal{L}}^{\dagger-1}(\mathbf{t}\mathbf{t}^T\mathbf{s})\right)_{\mathbf{x},t}$$

$$= -\left(\frac{\partial \boldsymbol{\mathcal{L}}^{-1}}{\partial m}\mathbf{u}, \boldsymbol{\lambda}\right)_{\mathbf{x},t} \quad \text{with} \quad \boldsymbol{\lambda} \equiv \boldsymbol{\mathcal{L}}^{\dagger-1}(\mathbf{t}\mathbf{t}^T\mathbf{s})$$

Here, $\boldsymbol{\lambda}$ is an adjoint field that satisfied the adjoint equation $\boldsymbol{\mathcal{L}}^{\dagger}\boldsymbol{\lambda} \equiv (\mathbf{t}\mathbf{t}^T\mathbf{s})$ with adjoint source $(\mathbf{t}\mathbf{t}^T\mathbf{s})$. The quantity $(\mathbf{t}\mathbf{t}^T\mathbf{s})$ is just the vector projection of $\mathbf{s}$ onto the direction $\mathbf{t}$. This is just what one might intuitively expect: that when the inner product p involves a component of the field, the adjoint field is just the kernel promoted to a vector that is parallel to the component.