The LU decomposition in action:
The “Fish Farm” problem

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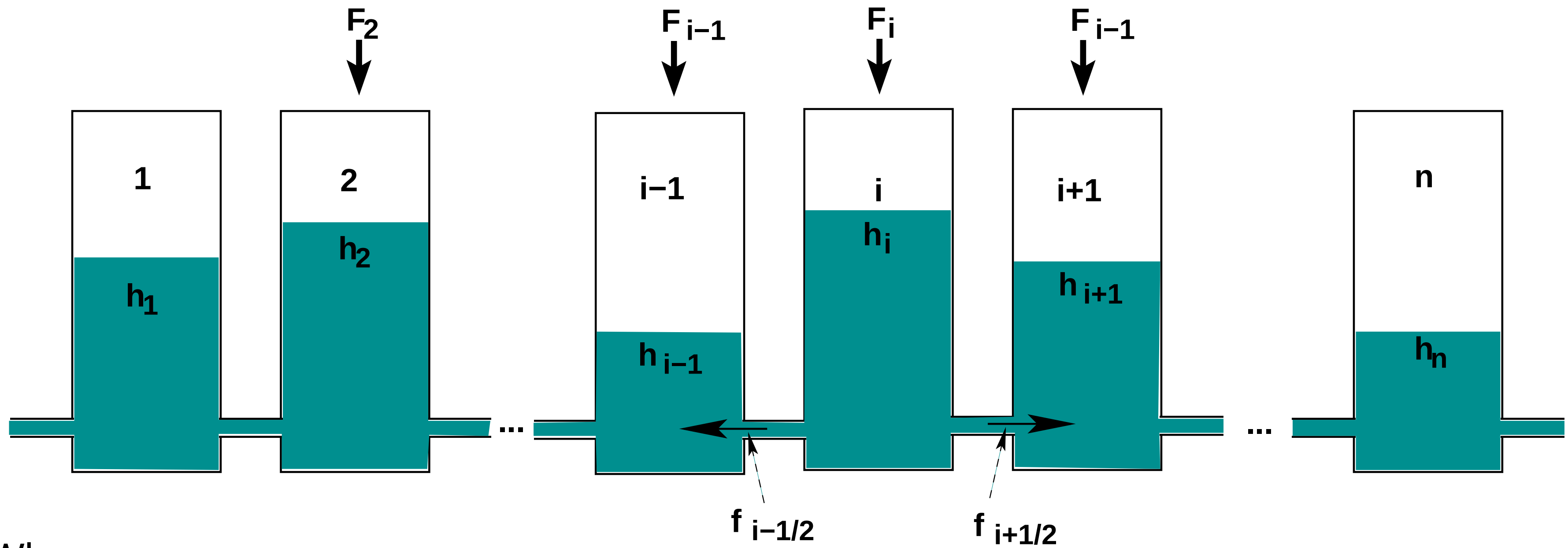
The "Fish Farm" Problem



Where

- h_i is the height of water in tank i
- F_i is the flux of water *added* to tank i
- and $f_{i+1/2} = -k(h_{i+1} - h_i)$ is the flux of water *between* tank i and $i+1$ (note: the flux is a signed quantity)

Point: Each tank is coupled to its two nearest neighbours. . . This is a very general problem that governs 1-D heat flow, Electric Potential, groundwater flow etc. (-$\frac{d}{dx}k\frac{dh}{dx} = F$)

Nearest neighbour coupling leads to a “Tridiagonal System” $A\mathbf{h} = \mathbf{r}$

Conservation of flux for tank i is

$$f_{i+1/2} - f_{i-1/2} = F_i$$

or

$$-k(h_{i+1} - h_i) + k(h_i - h_{i-1}) = F_i$$

or

$$-h_{i-1} + 2h_i - h_{i+1} = F_i/k = r_i$$

if k is constant

or as a system of equations (with h_1, h_n held fixed)

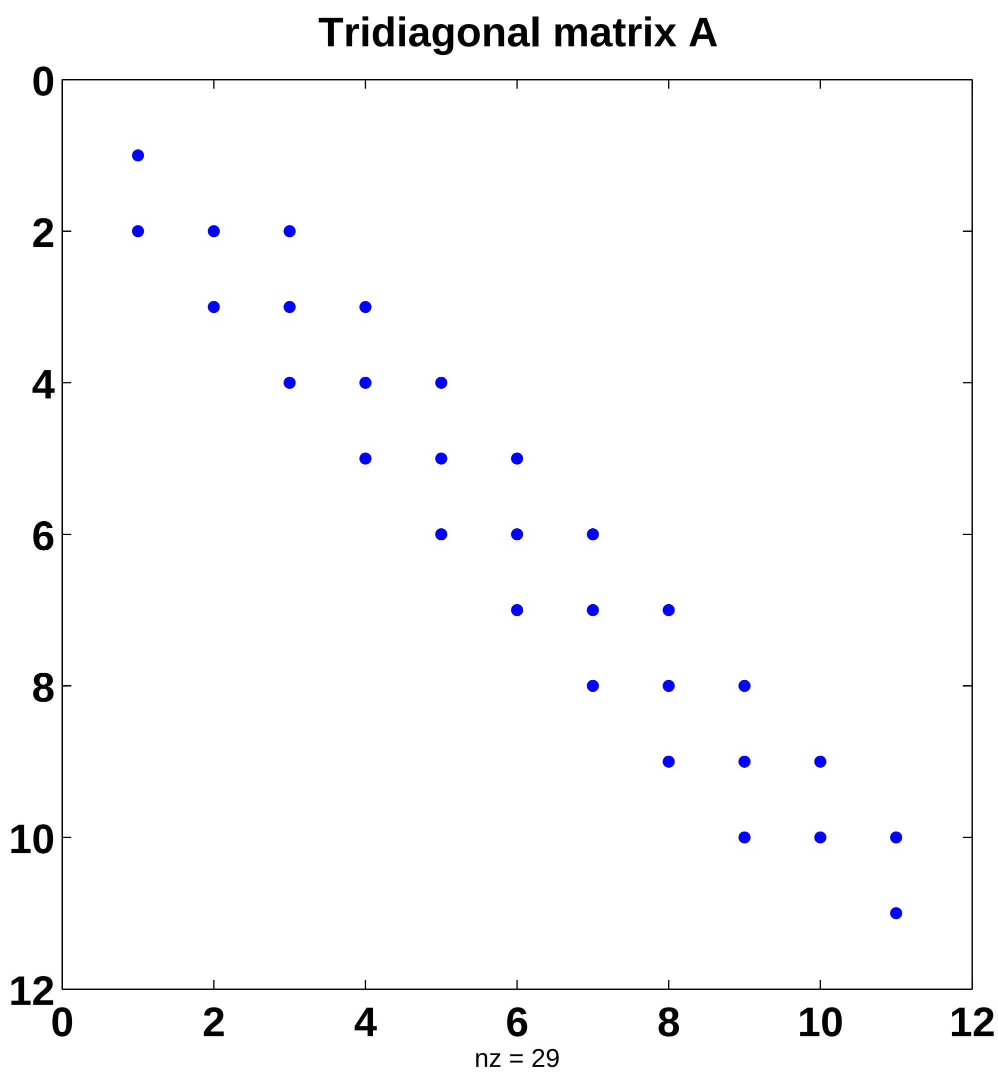
$$\begin{aligned}
h_1 &= c_1 \\
-h_1 + 2h_2 - h_3 &= r_2 \\
-h_2 + 2h_3 - h_4 &= r_3 \\
&\vdots \\
-h_{i-1} + 2h_i - h_{i+1} &= r_i \\
&\vdots \\
h_n &= c_n
\end{aligned}$$

or in matrix form as $A\mathbf{h} = \mathbf{r}$ where A is "Tridiagonal"

$$
\begin{bmatrix}
1 & & & & & & & & \\
-1 & 2 & -1 & & & & & & \\
 & -1 & 2 & -1 & & & & & \\
 & & & \ddots & & & & & \\
 & & & & \ddots & & & & \\
 & & & & & -1 & 2 & -1 & \\
 & & & & & & \ddots & & \\
 & & & & & & & \ddots & \\
 & & & & & & & & 1
\end{bmatrix}
\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_{i-1} \\ h_i \\ h_{i+1} \\ \vdots \\ h_n \end{bmatrix}
=
\begin{bmatrix} c_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{i-1} \\ r_i \\ r_{i+1} \\ \vdots \\ c_n \end{bmatrix}
$$

Note: (this is actually a discrete form of the ODE $-\frac{d^2h}{dx^2} = r$)

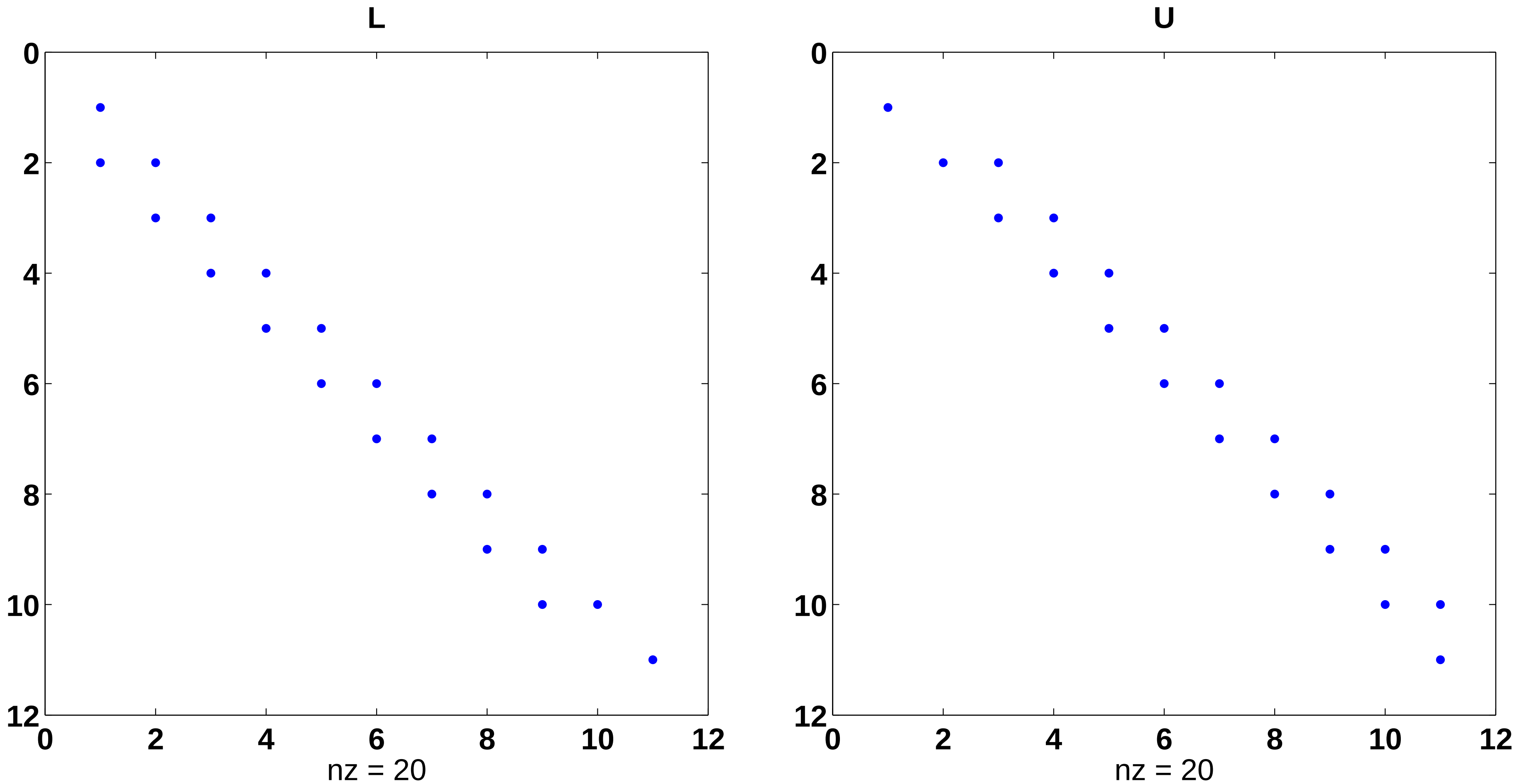
11 tanks yield an 11×11 sparse Matrix A



(in Matlab `spy(A)`)

LU decomposition of A

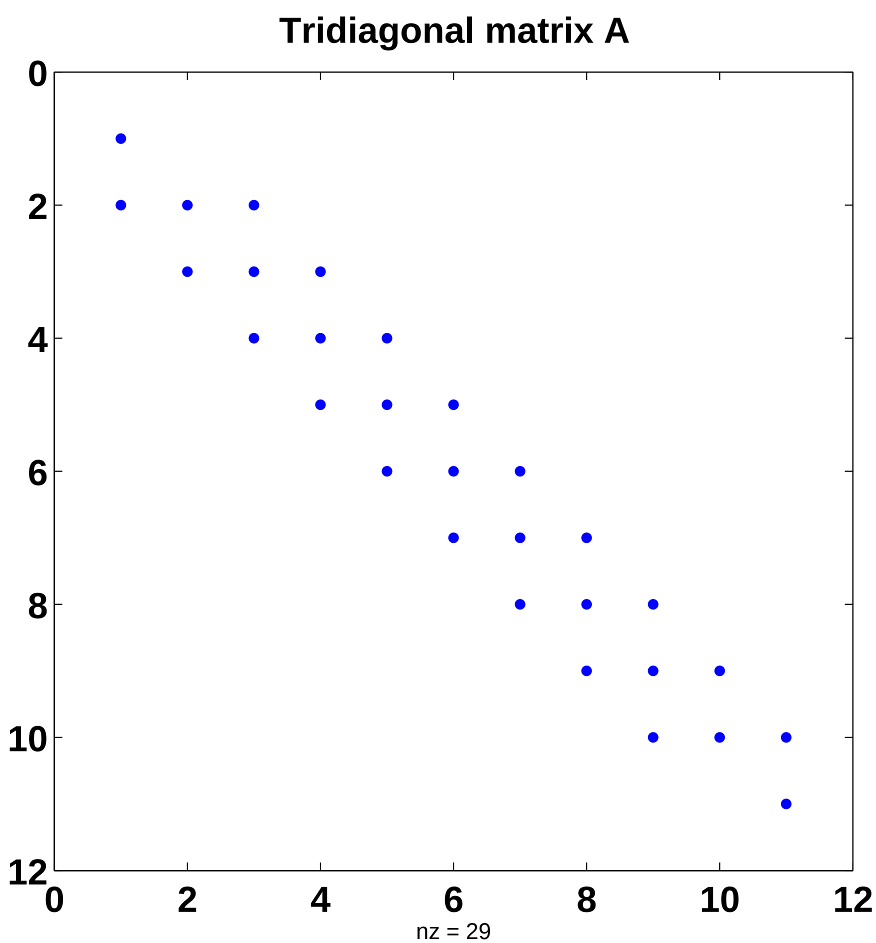
Matlab: `[L,U]=lu(A)`

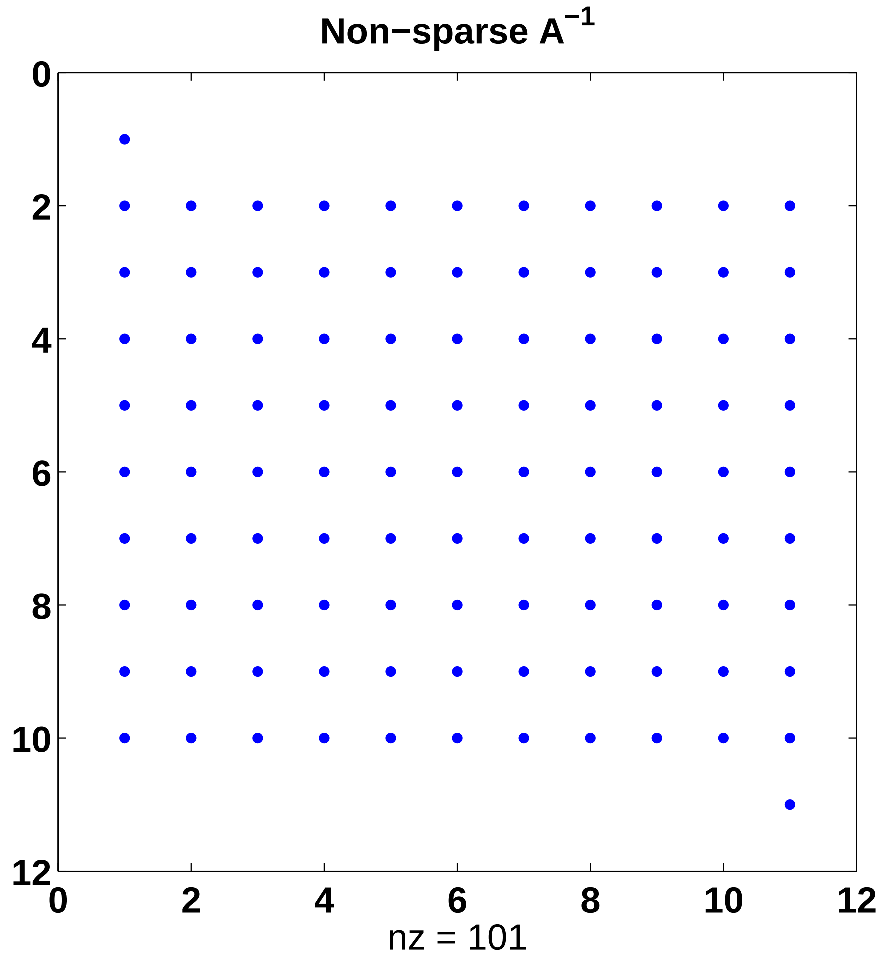


Note: The LU decomposition remains sparse for this matrix. . . why?

Comparison of A to A^{-1}

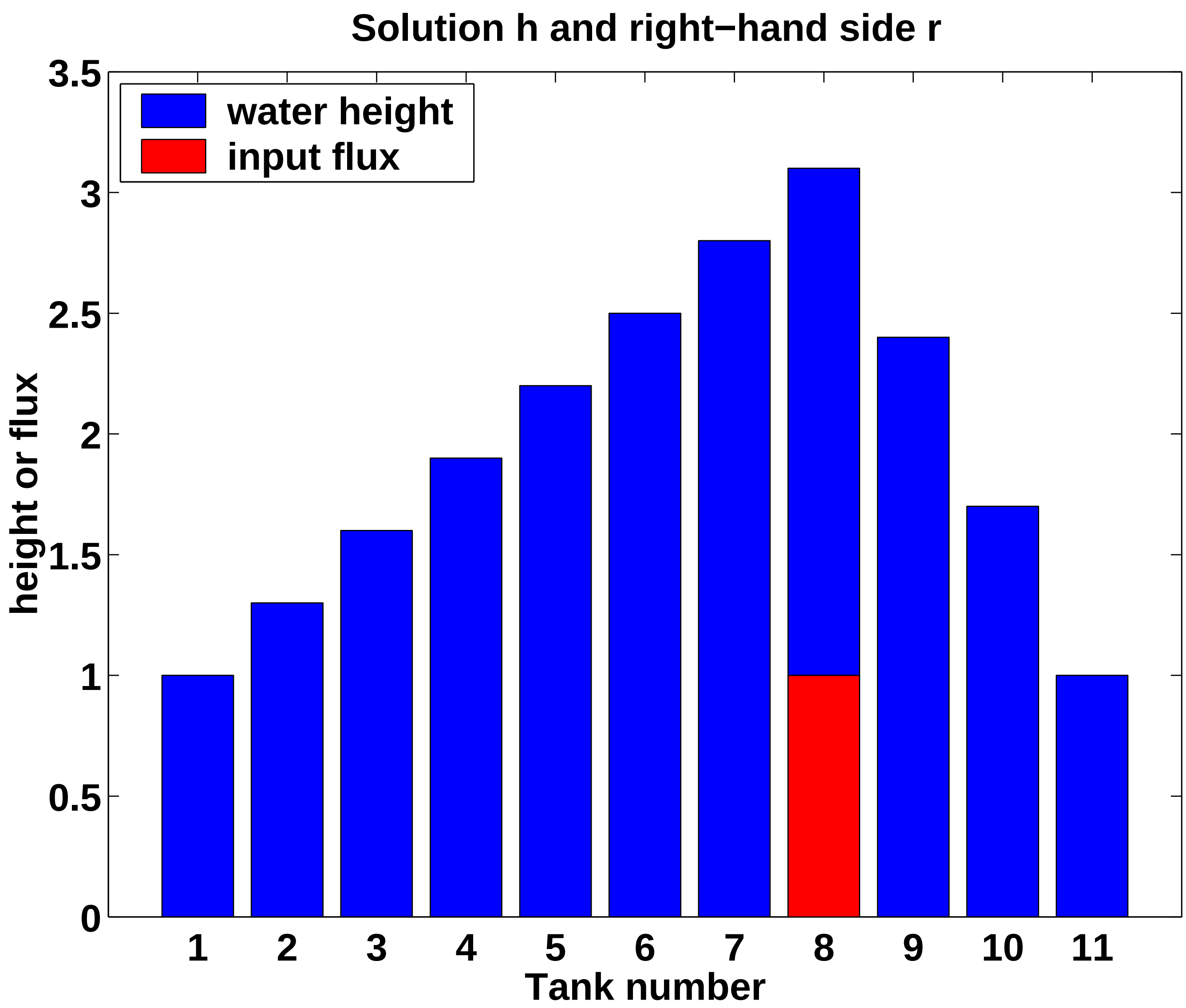
Matlab: `spy(A)` vs. `spy(inv(A))`



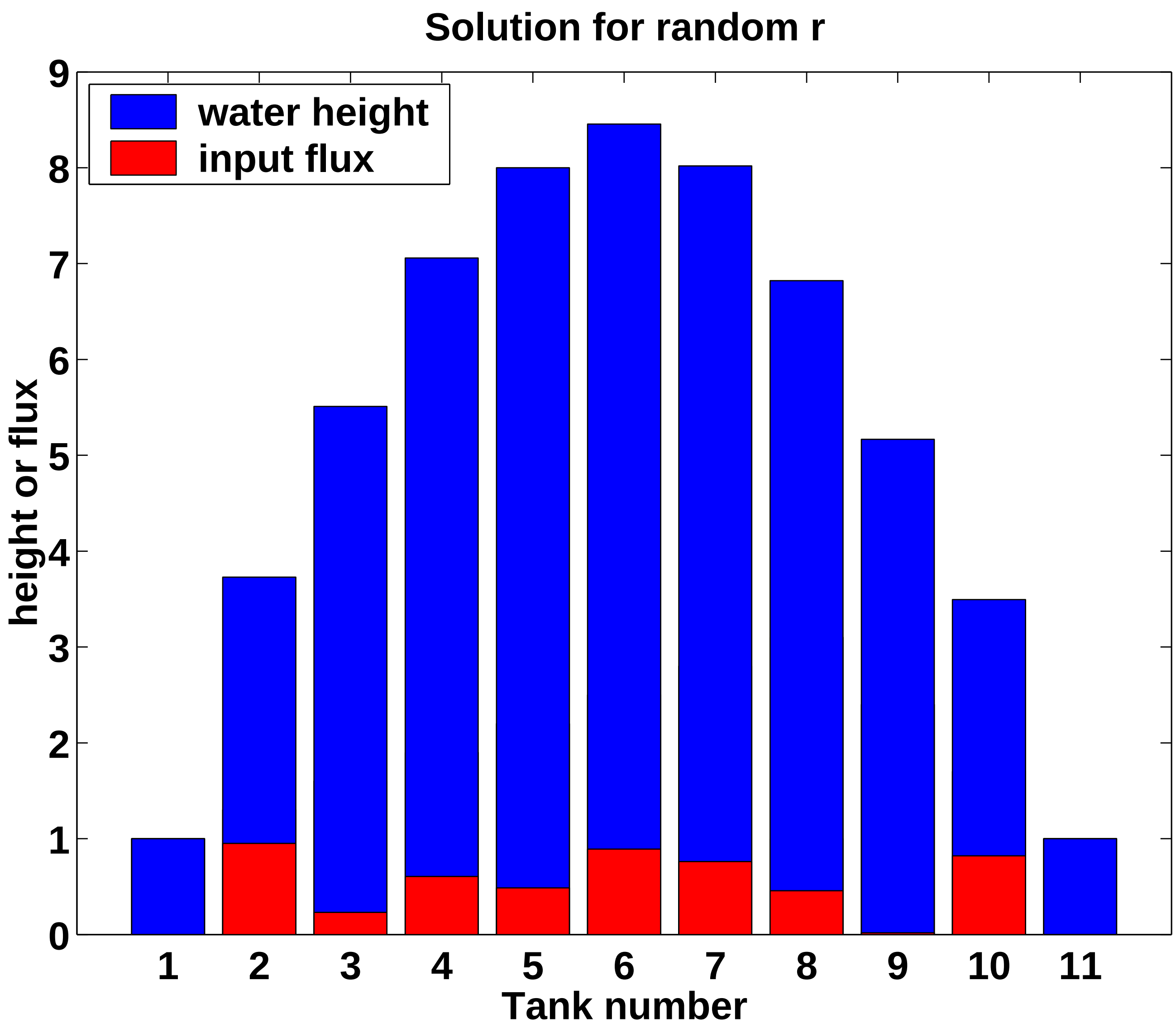


Solution of $LU\mathbf{h} = \mathbf{r}$

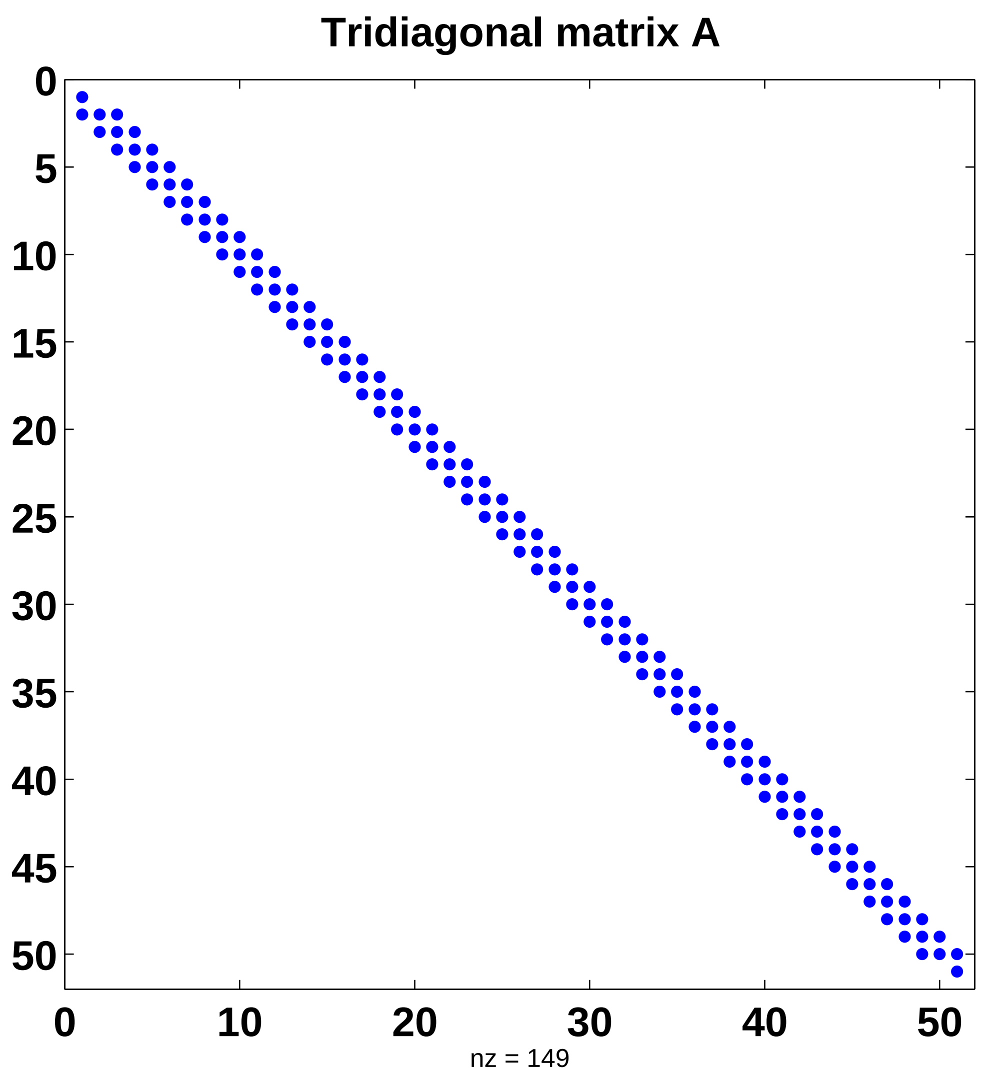
Matlab: `h=U\(L\r)`



Solution of LUh = r (random r)

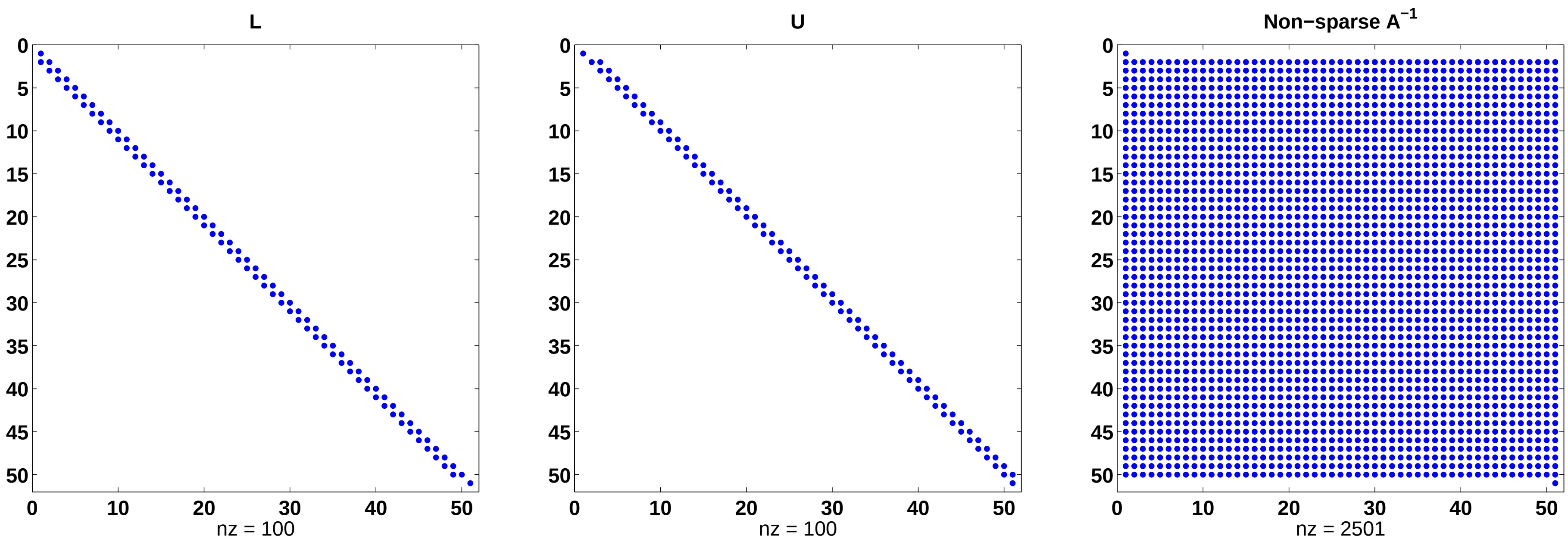


51 tanks: yield a 51×51 sparse Matrix A



(in Matlab `spy(A)`)

Comparison of LU to A^{-1}



Note: LU decomposition takes order N steps to solve $A\mathbf{h} = \mathbf{r}$ whereas $\mathbf{h} = A^{-1}\mathbf{r}$ takes order N^3 to just find A^{-1} and N^2 to multiply $A^{-1}\mathbf{r}$

Solution of $LU\mathbf{h} = \mathbf{r}$ for $N = 51$