42 Chapter 2 Solving Linear Equations

2 Solve the triangular system of Problem 1 by back substitution, y before x. Verify that x times (2, 10) plus y times (3, 9) equals (1, 11). If the right side changes to (4, 44), what is the new solution?

3 What multiple of equation 1 should be subtracted from equation 2?

\[ 2x - 4y = 6 \]
\[ -x + 5y = 0. \]

After this elimination step, solve the triangular system. If the right side changes to (−6, 0), what is the new solution?

4 What multiple \( \ell \) of equation 1 should be subtracted from equation 2?

\[ ax + by = f \]
\[ cx + dy = g. \]

The first pivot is \( a \) (assumed nonzero). Elimination produces what formula for the second pivot? What is \( y \)? The second pivot is missing when \( ad = bc \).

5 Choose a right side which gives no solution and another right side which gives infinitely many solutions. What are two of those solutions?

\[ 3x + 2y = 10 \]
\[ 6x + 4y = \]

6 Choose a coefficient \( b \) that makes this system singular. Then choose a right side \( g \) that makes it solvable. Find two solutions in that singular case.

\[ 2x + by = 16 \]
\[ 4x + 8y = g. \]

7 For which numbers \( a \) does elimination break down (1) permanently (2) temporarily?

\[ ax + 3y = -3 \]
\[ 4x + 6y = 6. \]

Solve for \( x \) and \( y \) after fixing the second breakdown by a row exchange.

8 For which three numbers \( k \) does elimination break down? Which is fixed by a row exchange? In each case, is the number of solutions 0 or 1 or \( \infty \)?

\[ kx + 3y = 6 \]
\[ 3x + ky = -6. \]
9 What test on \( b_1 \) and \( b_2 \) decides whether these two equations allow a solution? How many solutions will they have? Draw the column picture.

\[ 3x - 2y = b_1 \]
\[ 6x - 4y = b_2. \]

10 In the \( xy \) plane, draw the lines \( x + y = 5 \) and \( x + 2y = 6 \) and the equation \( y = \_ \_ \_ \) that comes from elimination. The line \( 5x - 4y = c \) will go through the solution of these equations if \( c = \_ \_ \_ \).

Problems 11–20 study elimination on 3 by 3 systems (and possible failure).

11 Reduce this system to upper triangular form by two row operations:

\[
\begin{align*}
2x + 3y + z &= 8 \\
4x + 7y + 5z &= 20 \\
-2y + 2z &= 0.
\end{align*}
\]

Circle the pivots. Solve by back substitution for \( z, y, x \).

12 Apply elimination (circle the pivots) and back substitution to solve

\[
\begin{align*}
2x - 3y &= 3 \\
4x - 5y + z &= 7 \\
2x - y - 3z &= 5.
\end{align*}
\]

List the three row operations: Subtract \( \_ \_ \_ \) times row \( \_ \_ \_ \) from row \( \_ \_ \_ \).

13 Which number \( d \) forces a row exchange, and what is the triangular system (not singular) for that \( d \)? Which \( d \) makes this system singular (no third pivot)?

\[
\begin{align*}
2x + 5y + z &= 0 \\
4x + dy + z &= 2 \\
y - z &= 3.
\end{align*}
\]

14 Which number \( b \) leads later to a row exchange? Which \( b \) leads to a missing pivot? In that singular case find a nonzero solution \( x, y, z \).

\[
\begin{align*}
x + by &= 0 \\
x - 2y - z &= 0 \\
y + z &= 0.
\end{align*}
\]

15 (a) Construct a 3 by 3 system that needs two row exchanges to reach a triangular form and a solution.

(b) Construct a 3 by 3 system that needs a row exchange to keep going, but breaks down later.
23 If you extend Problems 21–22 following the 1, 2, 1 pattern or the −1, 2, −1 pattern, what is the fifth pivot? What is the nth pivot?

24 If elimination leads to these equations, find three possible original matrices $A$:

\[
\begin{align*}
    x + y + z &= 0 \\
    y + z &= 0 \\
    3z &= 0.
\end{align*}
\]

25 For which two numbers $a$ will elimination fail on $A = \begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$?

26 For which three numbers $a$ will elimination fail to give three pivots?

\[
A = \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix}.
\]

27 Look for a matrix that has row sums 4 and 8, and column sums 2 and $s$:

Matrix $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

\[
\begin{align*}
    a + b &= 4 \\
    a + c &= 2 \\
    c + d &= 8 \\
    b + d &= s
\end{align*}
\]

The four equations are solvable only if $s = \ldots$. Then find two different matrices that have the correct row and column sums. Extra credit: Write down the 4 by 4 system $Ax = b$ with $x = (a, b, c, d)$ and make $A$ triangular by elimination.

28 Elimination in the usual order gives what pivot matrix and what solution to this "lower triangular" system? We are really solving by forward substitution:

\[
\begin{align*}
    3x &= 3 \\
    6x + 2y &= 8 \\
    9x - 2y + z &= 9.
\end{align*}
\]

29 Create a MATLAB command $A(2, :) = \ldots$ for the new row 2, to subtract 3 times row 1 from the existing row 2 if the matrix $A$ is already known.

30 Find experimentally the average first and second and third pivot sizes (use the absolute value) in MATLAB’s $A = \text{rand}(3, 3)$. The average of $\text{abs}(A(1, 1))$ should be 0.5 but I don’t know the others.
Problems 1–15 are about elimination matrices.

1. Write down the 3 by 3 matrices that produce these elimination steps:
   (a) $E_{21}$ subtracts 5 times row 1 from row 2.
   (b) $E_{32}$ subtracts 7 times row 2 from row 3.
   (c) $P$ exchanges rows 1 and 2, then rows 2 and 3.

2. In Problem 1, applying $E_{21}$ and then $E_{32}$ to the column $b = (1, 0, 0)$ gives $E_{32}E_{21}b = \underline{\underline{\text{_____}}}$.
   Applying $E_{32}$ before $E_{21}$ gives $E_{21}E_{32}b = \underline{\underline{\text{_____}}}$.
   When $E_{32}$ comes first, row \underline{\underline{_____}} feels no effect from row \underline{\underline{_____}}.

3. Which three matrices $E_{21}, E_{31}, E_{32}$ put $A$ into triangular form $U$?

   
   $$A = \begin{bmatrix}
   1 & 1 & 0 \\
   4 & 6 & 1 \\
   -2 & 2 & 0 \\
   \end{bmatrix}$$
   and $E_{32}E_{31}E_{21}A = U$.

   Multiply those $E$’s to get one matrix $M$ that does elimination: $MA = U$.

4. Include $b = (1, 0, 0)$ as a fourth column in Problem 3 to produce $[A \ b]$. Carry out the elimination steps on this augmented matrix to solve $Ax = b$.

5. Suppose $a_{33} = 7$ and the third pivot is 5. If you change $a_{33}$ to 11, the third pivot is \underline{\underline{_____}}. If you change $a_{33}$ to \underline{\underline{_____}}, there is no third pivot.

6. If every column of $A$ is a multiple of $(1, 1, 1)$, then $Ax$ is always a multiple of $(1, 1, 1)$. Do a 3 by 3 example. How many pivots are produced by elimination?

7. Suppose $E_{31}$ subtracts 7 times row 1 from row 3. To reverse that step you should \underline{\underline{_____}} 7 times row \underline{\underline{_____}} to row \underline{\underline{_____}}. This “inverse matrix” is $R_{31} = \underline{\underline{_____}}$.

8. Suppose $E_{31}$ subtracts 7 times row 1 from row 3. What matrix $R_{31}$ is changed into $I$? Then $E_{31}R_{31} = I$ where Problem 7 has $R_{31}E_{31} = I$. Both are true!

9. (a) $E_{21}$ subtracts row 1 from row 2 and then $P_{23}$ exchanges rows 2 and 3. What matrix $M = P_{23}E_{21}$ does both steps at once?
   (b) $P_{23}$ exchanges rows 2 and 3 and then $E_{31}$ subtracts row 1 from row 3. What matrix $M = E_{31}P_{23}$ does both steps at once? Explain why the $M$’s are the same but the $E$’s are different.

10. (a) What 3 by 3 matrix $E_{13}$ will add row 3 to row 1?
    (b) What matrix adds row 1 to row 3 and at the same time row 3 to row 1?
    (c) What matrix adds row 1 to row 3 and then adds row 3 to row 1?
Create a matrix that has \( a_{11} = a_{22} = a_{33} = 1 \) but elimination produces two negative pivots without row exchanges. (The first pivot is 1.)

Multiply these matrices:
\[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
-1 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
= 
\begin{bmatrix}
1 & 2 & 3 \\
1 & 3 & 1 \\
1 & 4 & 0
\end{bmatrix}.
\]

Explain these facts. If the third column of \( B \) is all zero, the third column of \( EB \) is all zero (for any \( E \)). If the third row of \( B \) is all zero, the third row of \( EB \) might not be zero.

This 4 by 4 matrix will need elimination matrices \( E_{21} \) and \( E_{32} \) and \( E_{43} \). What are those matrices?

\[
A = 
\begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{bmatrix}
\]

Write down the 3 by 3 matrix that has \( a_{ij} = 2i - 3j \). This matrix has \( a_{32} = 0 \), but elimination still needs \( E_{32} \) to produce a zero in the 3, 2 position. Which previous step destroys the original zero and what is \( E_{32} \)?

Problems 16–23 are about creating and multiplying matrices.

Write these ancient problems in a 2 by 2 matrix form \( Ax = b \) and solve them:

(a) \( X \) is twice as old as \( Y \) and their ages add to 33.

(b) \( (x, y) = (2, 5) \) and \( (3, 7) \) lie on the line \( y = mx + c \). Find \( m \) and \( c \).

The parabola \( y = a + bx + cx^2 \) goes through the points \( (x, y) = (1, 4) \) and \( (2, 8) \) and \( (3, 14) \). Find and solve a matrix equation for the unknowns \( (a, b, c) \).

Multiply these matrices in the orders \( EF \) and \( FE \) and \( E^2 \):

\[
E = 
\begin{bmatrix}
1 & 0 & 0 \\
a & 1 & 0 \\
b & 0 & 1
\end{bmatrix}
F = 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Also compute \( E^2 = EE \) and \( F^3 = FFF \).

Multiply these row exchange matrices in the orders \( PQ \) and \( QP \) and \( P^2 \):

\[
P = 
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
Q = 
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}.
\]

Find four matrices whose squares are \( M^2 = I \).
28 If \( AB = I \) and \( BC = I \) use the associative law to prove \( A = C \).

29 Choose two matrices \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) with \( \det M = ad - bc = 1 \) and with \( a, b, c, d \) positive integers. Prove that every such matrix \( M \) either has

EITHER row 1 \( \leq \) row 2 OR row 2 \( \leq \) row 1.

Subtraction makes \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} M \) or \( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} M \) nonnegative but smaller than \( M \). If you continue and reach \( I \), write your \( M \)'s as products of the inverses \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) and \( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \).

30 Find the triangular matrix \( E \) that reduces "Pascal's matrix" to a smaller Pascal:

\[
E = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 \\
1 & 3 & 3 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 2 & 1
\end{bmatrix}.
\]

Challenge question: Which \( M \) (from several \( E \)'s) reduces Pascal all the way to \( I \)?

### RULES FOR MATRIX OPERATIONS \( \mathbf{2.4} \)

I will start with basic facts. A matrix is a rectangular array of numbers or "entries." When \( A \) has \( m \) rows and \( n \) columns, it is an "\( m \) by \( n \)" matrix. Matrices can be added if their shapes are the same. They can be multiplied by any constant \( c \). Here are examples of \( A + B \) and \( 2A \), for 3 by 2 matrices:

\[
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
2 & 2 \\
4 & 4 \\
9 & 9
\end{bmatrix} = \begin{bmatrix}
3 & 4 \\
7 & 8 \\
9 & 9
\end{bmatrix} \quad \text{and} \quad 2 \begin{bmatrix}
1 & 2 \\
3 & 4 \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
2 & 4 \\
6 & 8 \\
0 & 0
\end{bmatrix}.
\]

Matrices are added exactly as vectors are—one entry at a time. We could even regard a column vector as a matrix with only one column (so \( n = 1 \)). The matrix \(-A\) comes from multiplication by \( c = -1 \) (reversing all the signs). Adding \( A \) to \(-A\) leaves the zero matrix, with all entries zero.

The 3 by 2 zero matrix is different from the 2 by 3 zero matrix. Even zero has a shape (several shapes) for matrices. All this is only common sense.

The entry in row \( i \) and column \( j \) is called \( a_{ij} \) or \( A(i, j) \). The \( n \) entries along the first row are \( a_{11}, a_{12}, \ldots, a_{1n} \). The lower left entry in the matrix is \( a_{n1} \) and the lower right is \( a_{nn} \). The row number \( i \) goes from 1 to \( m \). The column number \( j \) goes from 1 to \( n \).

Matrix addition is easy. The serious question is matrix multiplication. When can we multiply \( A \) times \( B \), and what is the product \( AB \)? We cannot multiply when \( A \) and \( B \) are 3 by 2. They don't pass the following test:

To multiply \( AB \): If \( A \) has \( n \) columns, \( B \) must have \( n \) rows.

If \( A \) has two columns, \( B \) must have two rows. When \( A \) is 3 by 2, the matrix \( B \) can be 2 by 1 (a vector) or 2 by 2 (square) or 2 by 20. Every column of \( B \) is ready to be multiplied by \( A \). Then \( AB \) is 3 by 1 (a vector) or 3 by 2 or 3 by 20.
those edges \((i \text{ to } k, k \text{ to } j)\) is missing. So the sum of \(a_{ik}a_{kj}\) is the number of 2-step paths leaving \(i\) and entering \(j\). Matrix multiplication is just right for this count.

The 3-step paths are counted by \(A^3\); we look at paths to node 2:

\[
A^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}
\]

counts the paths with three steps:

\[
\begin{array}{c|c|c}
\text{Path} & 1 \text{ to } 1 & 1 \text{ to } 2, 2 \text{ to } 1, 1 \text{ to } 2 \\
\hline
\text{Steps} & 1 \text{ to } 1 & 1 \text{ to } 2, 2 \text{ to } 1, 1 \text{ to } 2 \\
\end{array}
\]

These \(A^k\) contain the Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, 13, \ldots coming in Section 6.2. Fibonacci's rule \(F_{k+2} = F_{k+1} + F_k\) (as in 13 = 8 + 5) shows up in \((A)(A^k) = A^{k+1}:

\[
\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} = \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix} = A^{k+1}.
\]

There are 13 six-step paths from node 1 to node 1, but I can't find them all.

\(A^k\) also counts words. A path like 1 to 1 to 2 to 1 corresponds to the number 1121 or the word \(aaba\). The number 2 (the letter \(b\)) is not allowed to repeat because the graph has no edge from node 2 to node 2. The \(i, j\) entry of \(A^k\) counts the allowed numbers (or words) of length \(k+1\) that start with the \(i\)th letter and end with the \(j\)th.

The second graph also has diameter 2; \(A^2\) has no zeros.

**Problem Set 2.4**

Problems 1–17 are about the laws of matrix multiplication.

1. \(A\) is 3 by 5, \(B\) is 5 by 3, \(C\) is 5 by 1, and \(D\) is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

\[
BA \quad AB \quad ABD \quad DBA \quad A(B + C).
\]

2. What rows or columns or matrices do you multiply to find

(a) the third column of \(AB\)?

(b) the first row of \(AB\)?

(c) the entry in row 3, column 4 of \(AB\)?

(d) the entry in row 1, column 1 of \(CDE\)?

3. Add \(AB\) to \(AC\) and compare with \(A(B + C)\):

\[
A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}.
\]

4. In Problem 3, multiply \(A\) times \(BC\). Then multiply \(AB\) times \(C\).

5. Compute \(A^2\) and \(A^3\). Make a prediction for \(A^5\) and \(A^n\):

\[
A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}.
\]
2.4 Rules for Matrix Operations

(c) \( BA \) has rows 1 and 3 of \( A \) reversed and row 2 unchanged.

(d) All rows of \( BA \) are the same as row 1 of \( A \).

12 Suppose \( AB = BA \) and \( AC = CA \) for these two particular matrices \( B \) and \( C \):

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{commutes with} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]

Prove that \( a = d \) and \( b = c = 0 \). Then \( A \) is a multiple of \( I \). The only matrices that commute with \( B \) and \( C \) and all other 2 by 2 matrices are \( A = \) multiple of \( I \).

13 Which of the following matrices are guaranteed to equal \((A - B)^2\): \( A^2 - B^2 \), \((B - A)^2\), \( A^2 - 2AB + B^2 \), \( A(A - B) - B(A - B) \), \( A^2 - AB - BA + B^2 \)?

14 True or false:

(a) If \( A^2 \) is defined then \( A \) is necessarily square.

(b) If \( AB \) and \( BA \) are defined then \( A \) and \( B \) are square.

(c) If \( AB \) and \( BA \) are defined then \( AB \) and \( BA \) are square.

(d) If \( AB = B \) then \( A = I \).

15 If \( A \) is \( m \) by \( n \), how many separate multiplications are involved when

(a) \( A \) multiplies a vector \( x \) with \( n \) components?

(b) \( A \) multiplies an \( n \) by \( p \) matrix \( B \)?

(c) \( A \) multiplies itself to produce \( A^2 \)? Here \( m = n \).

16 To prove that \((AB)C = A(BC)\), use the column vectors \( b_1, \ldots, b_n \) of \( B \). First suppose that \( C \) has only one column \( c \) with entries \( c_1, \ldots, c_n \):

\( AB \) has columns \( Ab_1, \ldots, Ab_n \) and \( Bc \) has one column \( c_1b_1 + \cdots + c_nb_n \).

Then \((AB)c = c_1Ab_1 + \cdots + c_nb_n\) equals \( A(c_1b_1 + \cdots + c_nb_n) = A(BC)\).

Linearity gives equality of those two sums, and \((AB)c = A(BC)\). The same is true for all other \( c \) of \( C \). Therefore \((AB)C = A(BC)\).

17 For \( A = \begin{bmatrix} 3 & -1 \\ 2 & -2 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 6 \end{bmatrix} \), compute these answers and nothing more:

(a) column 2 of \( AB \)

(b) row 2 of \( AB \)

(c) row 2 of \( AA = A^2 \)

(d) row 2 of \( AAA = A^3 \).

Problems 18–20 use \( a_{ij} \) for the entry in row \( i \), column \( j \) of \( A \).

18 Write down the 3 by 3 matrix \( A \) whose entries are