Lecture 02:
Systems of Linear Equations and A<u>x</u>=<u>b</u>

Outline:

1) The relationship between systems of linear equations and Linear Combinations of vectors
2) The relationship between systems of equations and A<u>x</u>=<u>b</u>
3) Basic rules of Matrix-vector products
   A<u>x</u> as linear combinations of vectors
4) A<u>x</u>=<u>b</u>: the fundamental issues
5) Solving linear systems:
   Introduction to Gaussian Elimination

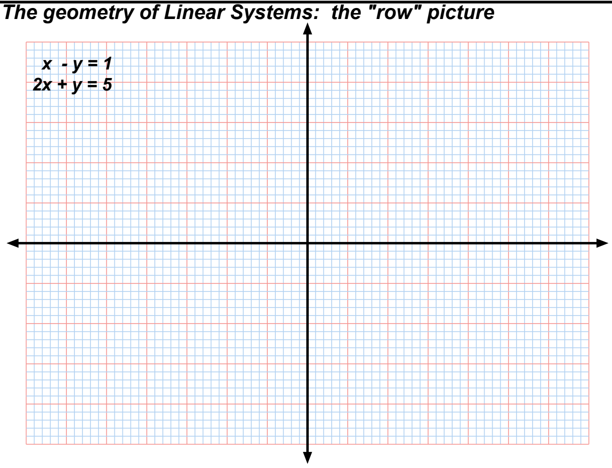
Systems of linear Equations

A simple example:

$$\begin{aligned} x - y &= 1 \\ 2x + y &= 5 \end{aligned}$$

Solution by "elimination" and "back-substitution":

The geometry of Linear Systems: the "row" picture



The geometry of Linear Systems: the "column" picture

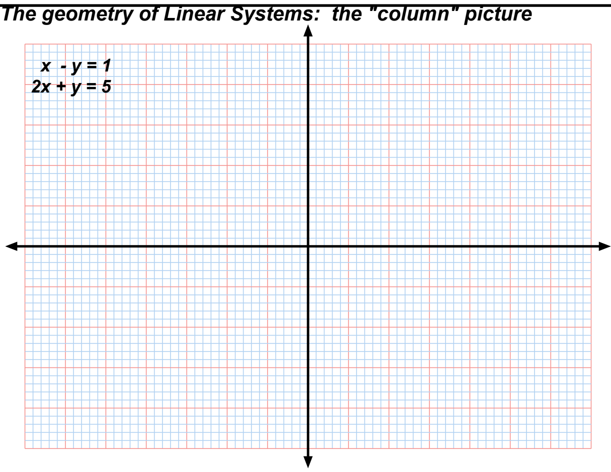
Linear systems of equations as Linear Combinations of vectors

$$\begin{aligned} x - y &= 1 \\ 2x + y &= 5 \end{aligned}$$

Is equivalent to:

Interpretation: solving linear systems is equivalent to finding linear combination of vectors that add up to another vector

The geometry of Linear Systems: the "column" picture



3x3 problems (3 equations, 3 unknowns):

A more compact approach:
Introducing the Matrix A

$$x - y = 1$$
$$2x + y = 5$$

2x2 problem:

3x3 problem:

Matrix-Vector Products: $A\underline{x}$

the Column Picture:

Example:

Matrix-Vector Products: $A\underline{x}$

the Row Picture (dot products):

Example:

Point!: Both Column and Row approaches are identical

Matrix Component Notation:

Matrices in Matlab:

Big Points:

1) $A\underline{x}$ is a linear combination of the columns of A
2) $A\underline{x}$ is a vector
3) $A\underline{x}$ maps $\underline{x}$ to a new vector $\underline{b}=A\underline{x}$

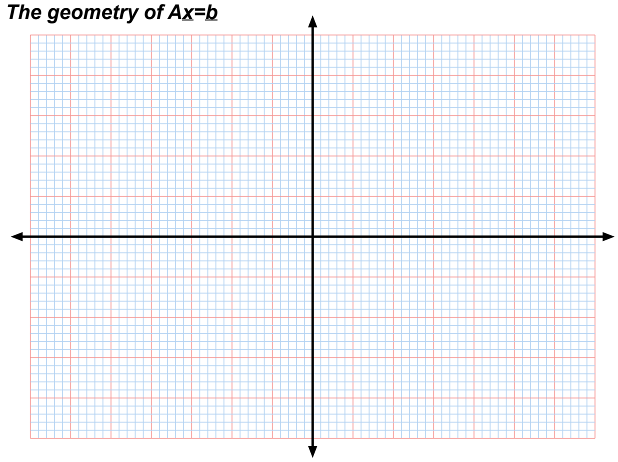
The Forward Problem: you know A and $\underline{x}$, just find $\underline{b}$.
Easy! Just use Matrix Vector multiplication. A solution must always exist.

The Inverse Problem: Given A and $\underline{b}$, find $\underline{x}$, such that $A\underline{x}=\underline{b}$
Much Harder! This is the problem of solving linear systems.

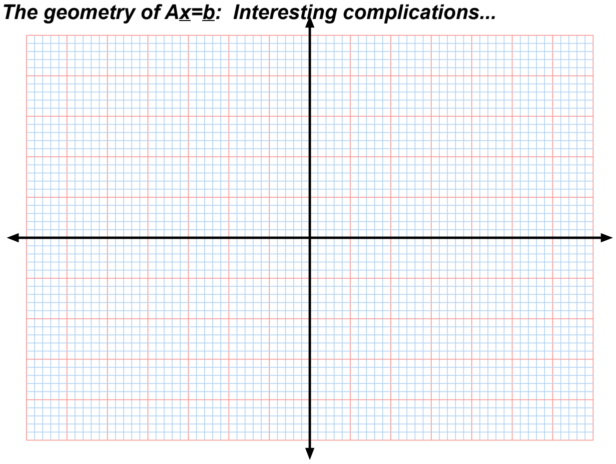
Fundamental Math questions: Given square A and a vector $\underline{b}$

1) ***does a solution $\underline{x}$ exist such that $A\underline{x}=\underline{b}$?***
2) ***is the solution $\underline{x}$ unique ?***
3) ***How do you find $\underline{x}$?***

The geometry of $A\underline{x}=\underline{b}$



The geometry of $A\underline{x}=\underline{b}$: Interesting complications...



General results (to be shown):
for square $n \times n$ A: $A\underline{x}=\underline{b}$ has exactly one of three outcomes

1) *There is a unique solution $\underline{x}$*
2) *There is no solution*
3) *There are an infinite number of solutions*

Question: How can you tell and how do you find the solution?

Gaussian Elimination:

A systematic algorithm to both diagnose and solve linear systems $A\underline{x}=\underline{b}$

Consider our toy 2x2 problem again:

$$\begin{aligned} x - y &= 1 \\ 2x + y &= 5 \end{aligned}$$

with solution: x=2, y=1

Suppose you wanted to eliminate x instead of y in 2nd equation?

Matrix Form of equations:

Gaussian Elimination:

A systematic algorithm to transform a general matrix A to upper Triangular form. $A\underline{x}=\underline{b} \rightarrow U\underline{x}=\underline{c}$

Upper Triangular system:

Easily solved by Back Substitution:

Gaussian Elimination:

A systematic algorithm to transform a general matrix A to upper Triangular form. $A\underline{x}=\underline{b} \rightarrow U\underline{x}=\underline{c}$

Need to transform both A and the RHS $\underline{b}$ to maintain equality.

Useful to Consider the Augmented matrix [A $\underline{b}$]

Idea of Elimination: Use row operations to zero out elements below the *pivot* using *multipliers* of the *pivot row*

Gaussian Elimination:

A systematic algorithm to transform a general matrix A to upper Triangular form. $A\underline{x}=\underline{b} \rightarrow U\underline{x}=\underline{c}$

Examples:

If it "works", Gaussian Elimination transforms A→U with n distinct pivots.

n Pivots implies a unique solution.

Failure of Gaussian Elimination:

Consider:

Temporary Failure of Gaussian Elimination:

Consider:

The Fix: Row exchange (permutation operation)

Gaussian Elimination on a 3x3 system of equations:

Consider:

$$\begin{aligned} 2x + y + 3z &= 3 \\ 4x + 3y + 8z &= 6 \\ -2x + 3z &= 1 \end{aligned}$$

Matrix Vector form:

Augmented Matrix form:

Gaussian Elimination on a 3x3 system of equations:

Example 2 (row exchange):

$$\begin{aligned} x + 2y + 4z &= 1 \\ 2x + 4y + 2z &= 2 \\ 6x + 10y - z &= 8 \end{aligned}$$

Matrix Vector form:

Augmented Matrix form:

Gaussian Elimination: The Overall Pattern

Linear combinations in 3 Dimensions