Lecture 03:
Systems of Linear Equations #2: Gaussian Elimination

Outline: Gaussian Elimination ($A\underline{x}=\underline{b} \rightarrow U\underline{x}=\underline{c}$)

1) Review mechanics
2) Review modes of failures
3) The general pattern of GE
3) GE using Matrices (E and P)
4) General Rules of Matrix-Matrix Operations
   Addition: (A+B)
   Matrix Multiplication: AB
   Matrix Powers: A^p

Gaussian Elimination on a 3x3 system of equations:

Example (with row exchange):

$$\begin{aligned} x + 2y + 4z &= 1 \\ 2x + 4y + 2z &= 2 \\ 6x + 10y - z &= 8 \end{aligned}$$

Matrix Vector form:

Augmented Matrix form:

Gaussian Elimination on a 3x3 system of equations:

Step 1:

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 2 & 4 & 2 & 2 \\ 6 & 10 & -1 & 8 \end{array}\right]$$

$A \qquad \underline{b}$

Step 2:

Step 3:

Step 4:

Failure of Gaussian Elimination (review):

Example 1:

Example 2:

The overall Pattern of Gaussian Elimination:

A recursive Algorithm that reduces $A\underline{x}=\underline{b}$ to $U\underline{x}=\underline{c}$

If successful: *U will contain n pivots on the diagonal (unique solution)*
If Fails: *U will contain at least 1 zero on the diagonal (no or ∞ solutions)*

Gaussian Elimination using matrices:

Two Basic operations:

1) *Elimination steps to zero out a_{ij}*
2) *Row exchanges to repair temporary failure*

BIG IDEA! *We can design Matrices (E and P) to do this for us*

Elementary Elimination Matrices E_{ij} :

$$\left[\begin{array}{ccc} 2 & 1 & 3 \\ 4 & 3 & 8 \\ -2 & 0 & 3 \end{array}\right]\left[\begin{array}{c} x \\ y \\ z \end{array}\right] = \left[\begin{array}{c} 3 \\ 6 \\ 1 \end{array}\right]$$

Elementary Elimination Matrices E_{ij} :

Matrix-Matrix Multiplication: C=AB

Column View:

Row View:

Matrix-Matrix Multiplication: C=AB

Other Examples: Diagonal Matrices
Left Multiplication: DA

Right Multiplication: AD

Does AD=DA (is matrix multiplication commutative?)

Gaussian Elimination:
a sequence of Elementary Elimination Matrices

Point: Now we're doing real Linear Algebra!
(Algebra of matrices and vectors, not arithmetic)

Permutation Matrices $P_{\alpha\beta\gamma\ldots}$:

Again: AB≠BA in general

Gaussian Elimination:
a sequence of Elementary Elimination Matrices

Point: Matrices Do things!

An important Digression:
General rules of Matrix-Matrix Operations

Matrix Shape:

An important Digression:
General rules of Matrix-Matrix Operations

Matrix Matrix Addition: A+B

Properties of Matrix Addition: (follow from scalar and vector addition)

An important Digression:
General rules of Matrix-Matrix Operations

Matrix Matrix Multiplication: C=AB

Examples (with numbers):

An important Digression:
General rules of Matrix-Matrix Operations

Properties of Matrix Multiplication:

An important Digression:
General rules of Matrix-Matrix Operations

Theorem: Matrix Mult is associative.
If A,B,C are matrices of appropriate shapes, then A(BC)=(AB)C

Proof (sketch): Show A(Bc)=(AB)c

Another important Digression:
Operation costs of Matrix-vector and Matrix-Matrix multiplication
(order matters!)

An important Digression:
General rules of Matrix-Matrix Operations

Matrix Powers: A^p

The matrix Inverse: A^{-1}

Summary:
Gaussian Elimination using Matrices