## Lecture 08:
### Vector Spaces and Subspaces #2: The Null Space N(A)

***Outline:***

1) Quick Review of the Column Space and Null Space
2) Finding N(A) part 1:
   Gaussian Elimination to *Upper Echelon Form*
   "Special Solutions"
   the rank *r* of a matrix
3) Finding N(A) part 2:
   Gauss-Jordan Elimination to *Reduced Row Echelon Form* (RREF)
   Examples
4) Existence and Uniqueness: The general solution to Ax=b

### *Review: The Column Space C(A)*

Definition:
The Column Space of a Matrix A is the vector subspace formed by all linear combinations of the columns of A (general mxn matrix)

1) C(A) *is* Ax when x assumes all values in $R^n$

2) C(A) is a subspace of R

3) C(A) controls the ______________________ of solutions to Ax=b

4) Ax=b will only have solutions iff b _________________________

### *Review: The Null Space N(A)*

Definition:
The Null Space of a Matrix A is the vector subspace formed by all solutions x to Ax=0 (i.e. is all linear combination of columns of A that cancel to zero)

1) N(A) is a subspace of R

2) N(A) controls the ______________________ of solutions to Ax=b

3) If A is invertible, then N(A)=______________

### *The Null Space: Algorithms for finding N(A)*

1) Inspired Guessing A = [1 2 ; 3 6 ]

2) Gaussian Elimination to solve Ax=0

(Only small issue is that U is no longer upper triangular)

### *The Null Space:*
#### *Algorithms for finding N(A): Gaussian Elimination to Upper Echelon Form*

A bigger example A = [1 1 2 1 ; 1 2 1 1 ]

Solve Ax=0 using Gaussian Elimination

### *Finding N(A): A closer look*

Gaussian Elimination (including row exchanges) transforms a general mxn matrix A to U which is now "upper echelon form"

1) Identify "Pivot" columns and "Free columns"
   (and associated Pivot variables and free variables)

2) Identify the *rank* of the matrix r = number of Pivot Columns

3) Identify the number of special solutions = number of free columns =

4) Find the Special Solutions: the linear combinations of Pivot Columns required to annihilate a single free column.

5) The Null Space is formed by *all linear combinations* of the special solutions

example A = [1 1 2 1 ; 1 2 1 1 ] again

## The Null Space: A better way

***Gauss-Jordan Elimination to Reduced Row Echelon Form R=rref(A)***

Again A = [1 1 2 1 ; 1 2 1 1 ]

1) take A to U By Gaussian Elimination

2) Continue by GJ Elimination (eliminate up then divide by the pivots) to R

3) when done:
   The pivot columns will be columns of the Identity Matrix
   Can read the Special Solutions right out of R

## The Null Space: A better way

***Gauss-Jordan Elimination to Reduced Row Echelon Form R=rref(A)***

Example A = [1 1 2 1 ; 1 2 1 1 ] goes to R = [ 1 0 3 1 ; 0 1 -1 0 ]

Point: N(R)=N( )

But Rx=0 is much easier to see

!! The Special solutions are the linear combinations of the pivot columns that annihilate each free column.

## The Null Space: A better way

***Gauss-Jordan Elimination to Reduced Row Echelon Form R=rref(A)***

Last Example: A = [ 1 2 2 5 ; 2 4 8 18 ; 3 6 10 23 ]

## The Null Space: A Trick

***Gauss-Jordan Elimination to Reduced Row Echelon Form R=rref(A)***

Last Example: A = [ 1 2 2 5 ; 2 4 8 18 ; 3 6 10 23 ]
R = [ 1 2 0 1 ; 0 0 1 2 ; 0 0 0 0 ]

Do it yourself Null Space:

## Last-Last one (Easy)

Last Example: A = [ 1 2 ; 3 4 ]

R = N(A)=

Point: All Invertible Matrices R=I, N(A)=Z

## *Putting it all together: The General Solution to $A\underline{x}=\underline{b}$*

Basic Approach:

1) use Gauss-Jordan Elimination to take [ A b ] to [ R d]

2) Find a Particular solution to $R\underline{x}_p=\underline{d}$
(combination of pivot columns and no free columns that add up to $\underline{d}$)

3) Find the special solutions to $R\underline{x}_N=\underline{0}$

4) The General solution: is $\underline{x}=\underline{x}_p+\underline{x}_N$

## *Putting it all together: The General Solution to $A\underline{x}=\underline{b}$*

Example:
A = [ 1 2 1 0 1; 2 4 1 0 0 ; 1 2 0 1 -4 ]
b = [ 1 1 1 ]