Lecture 10:
The Big picture cont'd

Outline:

1) Bases and Dimensions for the 4 Fundamental Subspaces of a mxn Matrix A
   - The Column Space C(A)
   - The Null Space N(A)
   - The Row Space $C(A^T)$
   - The Left Null Space $N(A^T)$
2) Orthogonality of the 4 subspaces
3) The full picture of Ax=b

Basis and Dimension of the 4 subspaces of a matrix A

Given A in R^{mxn} there are four fundamental subspaces associated with the matrix A

Name	Symbol	Subspace	Dimension
1) Column Space			
2) Null Space			
3) Row Space			
4) Left Null Space			

Basis and Dimension of the 4 subspaces of a matrix A

Example:

$$A=\begin{bmatrix}1&2&3&0&1\\1&2&3&1&3\\2&4&6&-2&-2\end{bmatrix} = \quad LR=\begin{bmatrix}1&0&0\\1&1&0\\2&-2&1\end{bmatrix}\begin{bmatrix}1&2&3&0&1\\0&0&0&1&2\\0&0&0&0&0\end{bmatrix}$$

The Column Space C(A): Controls _____________ of solutions to Ax=b

Subspace of

Dimension:

Basis:

Comments: C(A)___C(R)

Basis and Dimension of the 4 subspaces of a matrix A

Example:

$$A=\begin{bmatrix}1&2&3&0&1\\1&2&3&1&3\\2&4&6&-2&-2\end{bmatrix} = \quad LR=\begin{bmatrix}1&0&0\\1&1&0\\2&-2&1\end{bmatrix}\begin{bmatrix}1&2&3&0&1\\0&0&0&1&2\\0&0&0&0&0\end{bmatrix}$$

The Null Space C(A): Controls _____________ of solutions to Ax=b

Subspace of

Dimension:

Basis:

Comments: N(A)___N(R)

Basis and Dimension of the 4 subspaces of a matrix A

Example:

$$A=\begin{bmatrix}1&2&3&0&1\\1&2&3&1&3\\2&4&6&-2&-2\end{bmatrix}= \quad LR=\begin{bmatrix}1&0&0\\1&1&0\\2&-2&1\end{bmatrix}\begin{bmatrix}1&2&3&0&1\\0&0&0&1&2\\0&0&0&0&0\end{bmatrix}$$

The Row Space $C(A^T)$: Spanned by the rows of A (i.e. columns of A^T)

Subspace of

Dimension:

Basis:

Comments: $C(A^T)$___$C(R^T)$

Basis and Dimension of the 4 subspaces of a matrix A

Example:

$$A=\begin{bmatrix}1&2&3&0&1\\1&2&3&1&3\\2&4&6&-2&-2\end{bmatrix}= \quad LR=\begin{bmatrix}1&0&0\\1&1&0\\2&-2&1\end{bmatrix}\begin{bmatrix}1&2&3&0&1\\0&0&0&1&2\\0&0&0&0&0\end{bmatrix}$$

The Left Null Space $N(A^T)$: spanned by all solutions of $A^T\underline{y}=\underline{0}$ -or-
(Combinations of ______________ of A that cancel to zero)

Subspace of

Dimension:

Basis:

Comments: $N(A^T)$___$N(R^T)$

Basis and Dimension of the 4 subspaces of a matrix A

Find the dimension and a basis for each of the 4 subspaces: a recipe

1) Reduce A to R=rref(A)

2) Column Space: Get C(A) from R and A
dim C(A) = _____, basis:________________________

3) Null Space: Get N(A) from R
dim N(A) =______, basis:________________________

4) Row Space: Get $C(A^T)$ from R
dim $C(A^T)$ = ____, basis:________________________

5) Left Null Space: Get $N(A^T)$ the hard-way
dim($N(A^T)$)=_____, basis:_________________________

Last Example: a rank-1 matrix

$A=\underline{u}\,\underline{v}^T=\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1&2&1\end{bmatrix}$

What is the relationship between the 4 subspaces?

Orthogonality of the 4 subspaces

Point: For every matrix A there are 4 fundamental subspaces

Two in R^n:

Two in R^m:

Moreover: these subspaces are "*orthogonal complements*" such that

Orthogonal Subspaces: Orthogonal Complements

Definition:

Two subspaces S_1 and S_2 in R^k are "orthogonal complements" if

1) all vectors in S_1 are orthogonal to those in S_2
(i.e. the two subspaces only share the $\underline{0}$ vector)

2) dim(S1)+dim(S2) = k

Comment: any basis from S_1 together with any basis from S_2 form a complete basis for R^k. i.e. all vectors in R^k can be decomposed uniquely into a part in S_1 and S_2

Example:

$$A = \underline{u}\,\underline{v}^T = \begin{bmatrix} 1 \\ -1 \end{bmatrix} [\,1\ 2\ 1\,]$$

Orthogonality of the 4 subspaces: The Big (Strangian) Picture

The Big Picture of A$\underline{x}$=$\underline{b}$
Full row Rank:

The Big Picture of A$\underline{x}$=$\underline{b}$
Full Column Rank:

The Big Picture of A$\underline{x}$=$\underline{b}$
Full row and column Rank: Invertible Matrices