Lecture 13:
Orthonormal Bases and Q matrices
(towards the QR Factorization)

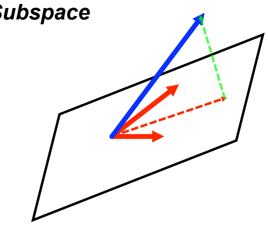
Outline:

1) Motivation: $A^TA\underline{x}=A^T\underline{b}$ as a projection problem using columns of A as a basis for C(A)

2) But some bases are better than others: *Orthonormal Bases*
   A) Definition and Q matrices ($Q^TQ=I$)
   B) Q matrices and Least-Squares problems
   B) Examples of Q matrices
   C) General Properties of Q Matrices
   D) Proof by basketball redux

3) Turning A to Q: Gram-Schmidt Othogonalization

Orthogonal Projection onto a Subspace



Orthonormal bases

Definition: a set of vectors $\underline{q}_1,\underline{q}_2,\ldots,\underline{q}_n$ in R^m (m>=n) are said to be orthonormal if

1) they are all unit vectors $\|\underline{q}_i\|=1$

2) they are all mutually orthogonal: $\underline{q}_i^T\underline{q}_j = 0$ if $i \neq j$

ultra compact definition $\underline{q}_i^T\underline{q}_j = \delta_{ij} =$

Orthonormal bases

Theorem: any set of orthogonal vectors is linearly independent (and therefore form a basis for span($\underline{q}_i$))

Proof: just use the fundamental definition of linear independence. the $\underline{q}_i$ are linearly independent if

$$c_1\underline{q}_1 + c_2\underline{q}_2 + \ldots c_n\underline{q}_n = \underline{0} \text{ iff all } c_i = 0$$

Q Matrices (Orthogonal Matrices)

Definition: a matrix whose columns consist of orthonormal vectors we'll called a Q matrix (Orthogonal/Unitary matrix if Q is square)

Comments:

1) ***the columns of Q form an orthonormal basis for C(Q)***
2) ***the fundamental property of Q matrices is Q^TQ=______***
3) ***If Q is square Q^T=____ and QQ^T = ____ (only true if square)***
4) ***In general QQ^T is a projection matrix onto C(Q)***

Q Matrices: Immediate practicality

Consider the overdetermined Least-Squares problem $A\underline{x}=\underline{b}$ if A=Q.

Point: in general A ≠ Q, however A can be transformed to Q via a new algorithm, which leads to a new Factorization A=QR...this is where we are going.

Q Matrices: Examples

Example #1: Permutation Matrices

Q Matrices: Examples

Example #2: Rotation Matrices

Q Matrices: Examples

Example #3: Householder reflection matrices $H=I-2\underline{u}\underline{u}^T$ ($||\underline{u}||=1$)

Q Matrices: Fundamental Properties

Property 1: Q matrices do not change the length of Vectors

Property 2: Q matrices preserve dot products

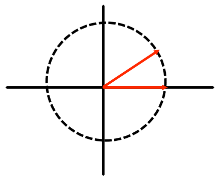
Comment: these properties make Q matrices very useful for numerical linear algebra

Q Matrices: Utility #2: Proof by basketball redux

Theorem: the dot product of any two unit vectors in R^n is

$$\underline{u}_1^T\underline{u}_2=\cos(\theta)$$

Proof: Part 1--dot product of simple vectors in R^2=cos(θ)



Part 2: The dot product is invariant to rotations

Q Matrices: Utility #2: Proof by basketball redux

Theorem: the dot product of any two unit vectors in R^n is

$$\underline{u}_1^T\underline{u}_2=\cos(\theta)$$

Proof: Part 3--Basketball time: The dot product of 2 unit vectors in R^n is equal to the dot product of 2 unit vectors in R^2 =cosθ

Comment: Idea of Change of basis is very powerful

Gram-Schmidt Orthogonalization:

A new algorithm for transforming an arbitrary basis into an orthonormal basis (or transforming A to Q)

Problem: given a set of vectors,

$\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$

that form a basis for some subspace S, fine an orthonormal basis

$q_1, q_2, \ldots, q_n$

for the same space.

Comments:

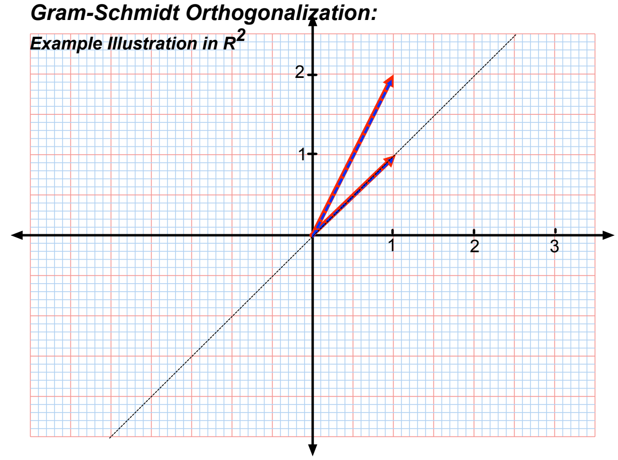
1) if $\underline{a}_i$ form the columns of a matrix A and span C(A), find an orthogonal matrix Q such that C(Q)=C(A)

2) An algorithm to do this (Gram-Schmidt Orthogonalization) is just another application of projections.

3) GS...is just a fancy name for "straightening out a bunch of vectors"

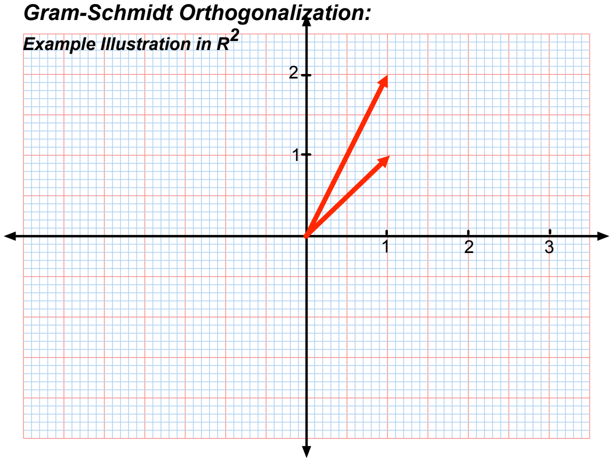
Gram-Schmidt Orthogonalization:

Example Illustration in R^2



Gram-Schmidt Orthogonalization:

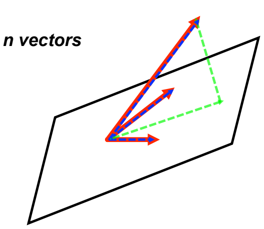
Example Illustration in R^2



Gram-Schmidt Orthogonalization:

Example in R^3: find an orthonormal basis for the C(A) where A = [1 1 1 ; 1 1 0; 1 0 0];

1) Extension of general algorithm to n vectors



Gram-Schmidt Orthogonalization:

Example in R^3: find an orthonormal basis for the C(A) where
A = [1 1 1 ; 1 1 0; 1 0 0];

2) Specific example

Gram-Schmidt Orthogonalization:

Example in R^3: find an orthonormal basis for the C(A) where
A = [1 1 1 ; 1 1 0; 1 0 0];

3) Comment Q not unique: suppose A=[1 1 1; 0 1 1 ; 0 0 1]