Lecture 14:
Gram-Schmidt Orthogonalization and the QR factorization

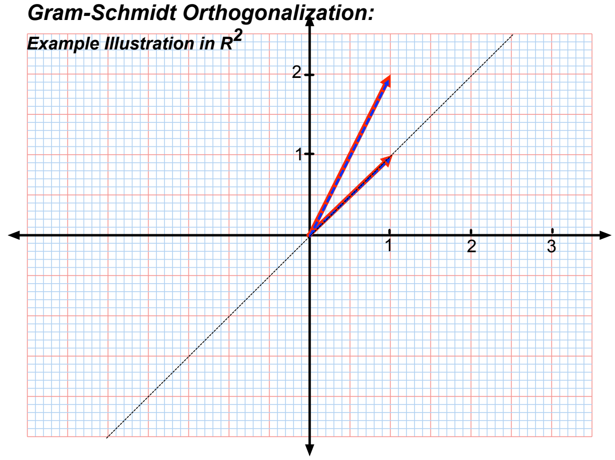
Outline:

1) Review GS for taking A to Q
   Examples galore
2) Gram-Schmidt orthogonalization leads to a new factorization
   A=QR
3) Using the QR to solve Linear Least-squares problems
4) Matlab
5) Computational issues, algorithms (modified Gram-Schmidt), computational costs...

6) Quick review of course so far (end of the middle)

Gram-Schmidt Orthogonalization:

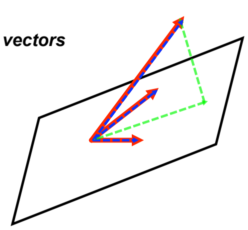
Example Illustration in R^2



Gram-Schmidt Orthogonalization:

Example in R^3: find an orthonormal basis for the C(A) where
A = [1 1 1 ; 1 1 0; 1 0 0];

1) Extension of general algorithm to n vectors



Gram-Schmidt Orthogonalization:

Example in R^3: find an orthonormal basis for the C(A) where
A = [1 1 1 ; 1 1 0; 1 0 0];

2) Specific example

Gram-Schmidt Orthogonalization:

Example in R^3: find an orthonormal basis for the C(A) where
A = [1 1 1 ; 1 1 0; 1 0 0];

3) Comment Q not unique: suppose A=[1 1 1; 0 1 1 ; 0 0 1]

Gram-Schmidt Orthogonalization:

Leads to a new Factorization A=QR

Analogy with Gaussian Elimination and PA=LU

1) Gaussian Elimination
- ***A) transforms A to U such that A=LU***
- ***B) you get L for free***
- ***C) Use the LU to solve square systems by solving $L\underline{c}=\underline{b}$, $U\underline{x}=\underline{c}$.***
- ***D) Matlab implimentation is just x=A\b***

2) Gram-Schmidt Orthogonalization
- ***A) transforms A to Q such that A=QR***
- ***B) and you get R for free (but I need to explain R)***
- ***C) Use the QR to solve linear least squares problems using $R\hat{\underline{x}}=Q^T\underline{b}$***
- ***D) Matlab implementation is just x=A\b***

Gram-Schmidt Orthogonalization:

Leads to a new Factorization A=QR

What is the R in A=QR?

1) Generally: R=______

2) R describes how the columns of A can be written (uniquely) in terms of the columns of Q.

Gram-Schmidt Orthogonalization:

Leads to a new Factorization A=QR

What is the R in A=QR?

3) Because of GS algorithm R is Upper Triangular! (Nice!)

Gram-Schmidt Orthogonalization:

Leads to a new Factorization A=QR

What is the R in A=QR?

4) Upper Triangular R, makes over-determined least-squares problem $A\underline{x}=\underline{b}$, reduce to the Triangular problem $R\hat{\underline{x}}=Q^T\underline{b}$. This can be solved quickly by back substitution.

Gram-Schmidt Orthogonalization:

Leads to a new Factorization A=QR

What is the R in A=QR?

5) Comment: only least squares solution $\hat{\underline{x}}$, needs R

Projection $\underline{p}$=__________________

Projection Matrix P=_______ (error $\underline{e}$=__________)

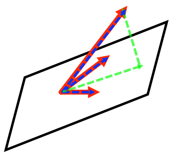
Alternative proof: let $P=A(A^TA)^{-1}A^T$ and A=QR

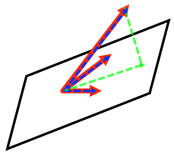
QR Factorization and Applications

Example: Find best fit straight line through (0,0), (1,2),(2,1) by using the Normal Equations and QR...

QR Factorization: Computational issues

Computing the QR: a simple algorithm Modified Gram-Schmidt





QR Factorization: Computational issues

Computing the QR: a simple algorithm Modified Gram-Schmidt

Computational Costs: Normal Equations vs QR

QR Factorization: Computational issues

Computing the QR: Other Algorithms: A to R by Householder Reflections

Where We are...a mini-course review

Part I)

- *A) Principal Equation:* ______________
- *B) Algorithm:* ______________
- *C) Factorization:* ______________
- *D) Applications:* ______________
- *E) Theory:* ______________

Part II)

- *A) Principal Equation:* ______________
- *B) Algorithm:* ______________
- *C) Factorization:* ______________
- *D) Applications:* ______________
- *E) Theory:* ______________