# Chapter 1

Conservation Equations

Nearly all of the modeling of physical phenomena is based on the simple statement "You can't get something for nothing" i.e. there are certain important physical properties that must be conserved. This section presents a generic recipe for deriving conservations equation of all kinds and will demonstrate the physical basis of most of the frequently occurring terms in partial (and ordinary) differential equations. When we are finished, we should, with a bit of thought, be able to formulate any quantitative problem in continuum mechanics.

1.1   Conservation of anything

Integral Form   Consider an arbitrary inertial frame in space of volume V enclosed by a surface S i.e.
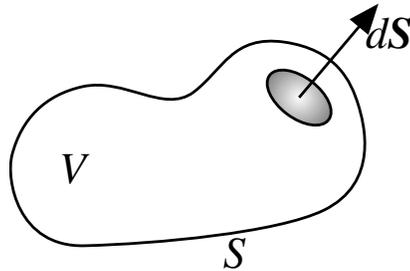
Here, $d\mathbf{S}$ is the vector normal to a small patch on the surface S. This vector points outwards by convention.

If we now consider how any quantity Φ (in units of stuff per unit volume) can change within this volume, the only way to change the amount of Φ with time is to flux it through the boundary or create it within the volume. If we let $\mathbf{F}$ be the flux of Φ in the absence of fluid transport (e.g. heat conduction), $\Phi\mathbf{V}$ be the transport flux (stuff per unit area per unit time) and H be a source or sink of Φ then the statement of conservation of Φ for the volume V becomes

$$\frac{d}{dt}\int_V \Phi dV = -\int_S \mathbf{F}\cdot d\mathbf{S} - \int_S \Phi\mathbf{V}\cdot d\mathbf{S} + \int_V H dV \qquad (1.1.1)$$

The negative signs in from of the surface integrals are present because a positive outward flux corresponds to a negative rate of change of the integral on the left

side of Eq. (1.1.1). This equation is always true, independent of the size of the blob and even if the fields are not continuous; however, because of the integrals, any information on the spatial structure of the fields on a scale smaller than the "blob" size is lost.

Equations for averaged properties: ODE's and box models   This loss of spatial information is not always a bad thing and sometimes we are only interested in the changes in the *average* properties in the blob with time. An example would be the mean concentration of a tracer in an ocean basin or "geochemical reservoir". Equations for average properties, however, are readily derived from Eq. (1.1.1).

First we define the volume average of a function f as

$$\bar{f} = \frac{1}{V} \int_V f \, dV \tag{1.1.2}$$

Next we will simply lump all the flux terms into two terms i.e. the stuff coming in minus the stuff coming out

$$\int_S [\mathbf{F} + \Phi\mathbf{V}] \cdot d\mathbf{S} = \dot{M}_{out} - \dot{M}_{in} \tag{1.1.3}$$

where $\dot{M}$ has units of stuff per unit time. Using these definitions and dividing Eq. (1.1.1) by the volume V yields

$$\frac{d\bar{\Phi}}{dt} = \dot{\Phi}_{in} - \dot{\Phi}_{out} + \bar{H} \tag{1.1.4}$$

where $\dot{\Phi} = \dot{M}/V$.

Equation (1.1.4) is an *Ordinary differential equation* for changes in the average volume density of Φ with time. All information about the spatial variation of Φ within the volume, the fluxes or the sources has been removed but this approach is often good enough for government work or *Box Models* if only averaged properties are necessary. If the functional form of the sources and fluxes are known, Eq. (1.1.4) can often be solved analytically, however, quite often the fluxes depend on the concentrations, can be strongly non-linear or there are many substances to solve for simultaneously. Chapter 4 will deal with numerical solutions of systems of ODE's.

Equations for locally continuous fields: PDE's   If we're interested in spatial variations as well as time, we need to subdivide into many smaller blobs. How small is small? In continuum mechanics, we have the concept of the *Representative Volume Element* or RVE (or useful blob size). Given some spatial field Φ, the scale at which the RVE is defined is determined by several properties

1. Φ is relatively constant on a scale comparable to the RVE, i.e. the average value of $\bar{\Phi}$ defined for the RVE is a good approximation to Φ anywhere in the RVE.

2. the average of Φ for each contiguous RVE varies smoothly i.e. $\bar{\bar{\Phi}}$ is differentiable (e.g. $\nabla\Phi$ makes sense at the scale of the RVE).

Caveats: Once we presume that there is a scale where the RVE is well defined we are also assuming that

1. when we discuss the variation of Φ in space we really are talking about the average of Φ i.e. $\bar{\bar{\Phi}}$ defined for the RVE

2. We are not interested in anything smaller than this scale

3. Any variation smaller than this scale does not change the gross behaviour of the problem.[1]

Given the existence of a suitable continuum length scale, we can now rewrite Eq. (1.1.1) as a local partial differential equation. Because the property of interest is differentiable, we can replace the surface integrals in Eq. (1.1.1) using Gauss' theorem

$$-\int_S \mathbf{F} \cdot d\mathbf{S} - \int_S \Phi\mathbf{V} \cdot d\mathbf{S} = -\int_V \nabla \cdot (\mathbf{F} + \Phi\mathbf{V}) dV \qquad (1.1.5)$$

Moreover, because the surface and volume are fixed in an inertial frame then the time derivative of the summed properties is equal to the sum of the local time derivatives or

$$\frac{d}{dt}\int_V \Phi dV = \int_V \frac{\partial\Phi}{\partial t} dV \qquad (1.1.6)$$

Substituting Eqs. (1.1.5)–(1.1.6) into (1.1.1) yields

$$\int_V \left[\frac{\partial\Phi}{\partial t} + \nabla \cdot (\mathbf{F} + \Phi\mathbf{V}) - H \right] dV = 0 \qquad (1.1.7)$$

Because V is of arbitrary shape and size, Eq. (1.1.7) can only be satisfied if the term in square brackets is zero everywhere (or at least for every RVE), therefore

$$\frac{\partial\Phi}{\partial t} + \nabla \cdot (\mathbf{F} + \Phi\mathbf{V}) - H = 0 \qquad (1.1.8)$$

This is the general form which all conservation laws take in continuum mechanics.

1.2 General conservation of mass, energy and momentum

Given Eq. (1.1.8) for the conservation of anything, it is now straightforward to consider conservation of the 3 most important quantities, mass, energy and momentum (force balance).

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[1]Quite often, problems in Earth Sciences violate caveats 2 and 3 yet we persevere by introducing "sub-grid" parameterizations of small scale processes or produce combined "micro-macro" models that try to patch together important small scale processes and the large scale dynamics.

Conservation of Mass   To derive conservation of mass we just substitute $\Phi = \rho$ (density is the amount of mass per unit volume), $\mathbf{F} = 0$ (mass flux can only change due to transport) and $H = 0$ (mass cannot be created or destroyed) into Eq. (1.1.8) to get

$$\frac{\partial \rho}{\partial t} + \mathbf{\nabla} \cdot (\rho \mathbf{V}) = 0 \tag{1.2.1}$$

This equation is often referred to as *the continuity equation*.

Conservation of Energy (heat)   For a single phase material, the amount of heat per unit volume is $\Phi = \rho c_P T$ where c_P is the specific heat (energy per unit mass per degree Kelvin) at constant pressure and T is the temperature. The heat flux has two components due to conduction and transport. In the absence of transport the heat flux is $\mathbf{F} = -k\mathbf{\nabla}T$ where k is the *thermal conductivity*. Note that heat flows opposite to $\mathbf{\nabla}T$, i.e. heat flows from hot to cold. The transport flux is $\rho c_P T \mathbf{V}$. Finally, unlike mass, heat can be created in a region due to terms like radioactive decay or viscous dissipation and shear heating. We will just lump all the source terms into H. Thus the simplest conservation of heat equation is

$$\frac{\partial \rho c_P T}{\partial t} + \mathbf{\nabla} \cdot (\rho c_P T \mathbf{V}) = \mathbf{\nabla} \cdot k\mathbf{\nabla}T + H \tag{1.2.2}$$

For constant c_P and k, this equation can also be rewritten using Equation (1.2.1) as

$$\frac{\partial T}{\partial t} + \mathbf{V} \cdot \mathbf{\nabla}T = \kappa \nabla^2 T + H/\rho c_P \tag{1.2.3}$$

Where $\kappa = k/\rho c_P$ is the *thermal diffusivity* with units m^2s^{-1}. Note: terms that look like

$$\frac{D_\mathbf{V}T}{Dt} = \frac{\partial T}{\partial t} + \mathbf{V} \cdot \mathbf{\nabla}T \tag{1.2.4}$$

are known as the *material derivative* and can be shown to be the change in time of some property (here temperature) as observed in a frame moving with at velocity $\mathbf{V}$ (we will show this explicitly in Section 5.2).

Conservation of Momentum   Conservation of momentum or force balance can be derived in exactly the same way, however momentum is a vector field (not a scalar field like temperature). In general momentum is $m\mathbf{V}$, therefore the amount of momentum per unit volume is $\Phi = \rho\mathbf{V}$. Other than advecting momentum, the only other way to change the momentum in our RVE is to exert forces on it. These forces come in two flavors. First, there is the stress that acts on the surface of the volume with local force $\mathbf{f} = \boldsymbol{\sigma} \cdot d\mathbf{S}$. As stress is simply force per unit area, the stress can also be thought of as a flux of force or $\mathbf{F} = -\boldsymbol{\sigma}$ (the negative sign insures that if the net force on the volume points in, the momentum increases). The second force acting on the volume are any body forces such as gravity. The body force acts a source of momentum, thus $H = \rho\mathbf{g}$ where $\mathbf{g}$ is the net acceleration. Substituting into Eq. (1.1.8) yields conservation of momentum

$$\frac{\partial \rho\mathbf{V}}{\partial t} + \mathbf{\nabla} \cdot (\rho\mathbf{V}\mathbf{V}) = \mathbf{\nabla} \cdot \boldsymbol{\sigma} + \rho\mathbf{g} \tag{1.2.5}$$

This equation can also be derived (perhaps more simply) by considering the continuous form of Newton's Law $\mathbf{f} = m\mathbf{a}$. The easiest way to understand Eq. (1.2.5) is to think in terms of each of the three cartesian components of the linear momentum which must be conserved individually. Using index notation, Eq. (1.2.5) can be written for the ith component of the momentum as

$$\frac{\partial \rho V_i}{\partial t} + \frac{\partial}{\partial x_j}\left[\rho V_i V_j\right] = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i \tag{1.2.6}$$

where $i = 1, 2, 3$ and summation is assumed over $j = 1, 2, 3$. Using conservation of mass, Eq. (1.2.5) can also be written

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \boldsymbol{\nabla})\mathbf{V} = \frac{1}{\rho}\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \mathbf{g} \tag{1.2.7}$$

Note that the advection of momentum $(\mathbf{V} \cdot \boldsymbol{\nabla})\mathbf{V}$ is non-linear and this is the term that leads to much of the interesting behaviour in fluid mechanics.

1.3 Constitutive relations and approximations

Equations (1.2.1), (1.2.2) and (1.2.6) are applicable to any continuum. To complete the equations, however, requires some additional constraints that relate stress to velocity (strainrate) or displacement, as well as any thermodynamic equations of state for material properties such as heat capacity, density or conductivity (although these are often assumed to be constant).

Viscous fluids The simplest rheology for a fluid is that of an isotropic incompressible fluid where the stress is

$$\sigma_{ij} = -P\delta_{ij} + \eta\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) \tag{1.3.1}$$

where P is the fluid pressure, η is the shear viscosity, and the final bracket on the right hand side is the strain-rate tensor $\dot{\epsilon}_{ij}$. Substituting into (1.2.7) for a constant viscosity fluid gives the Navier-Stokes equation

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \boldsymbol{\nabla})\mathbf{V} = -\frac{1}{\rho}\boldsymbol{\nabla}P + \nu\nabla^2\mathbf{V} + \mathbf{g} \tag{1.3.2}$$

where $\nu = \eta/\rho$ is the dynamic viscosity.

Elastic bodies Conservation of momentum also applies to elastic rheologies. Here however the stress is proportional to displacement not velocity. Hooke's law for a general linear elastic body relates stress to strain by

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \tag{1.3.3}$$

where C_{ijkl} is a fourth order tensor. For many problems, however, we assume the medium is locally *isotropic* and Hooke's law reduces to

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij} \tag{1.3.4}$$

where

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.3.5}$$

is the strain tensor, μ is the shear modulus and λ is the bulk modulus. **u** are the elastic displacements from equilibrium. In an elastic medium, the position of a particle is $\mathbf{x} = \mathbf{x}_0 + \mathbf{u}$ and the displacements are assumed small. The velocity of a particle is therefore

$$\mathbf{V} = \frac{\partial \mathbf{x}}{\partial t} = \frac{\partial \mathbf{u}}{\partial t} \tag{1.3.6}$$

Substituting into Eq. (1.2.7) and assuming that the advective terms are small (they are order $\mathbf{u} \cdot \mathbf{u}$ which is very small) yields

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \frac{\mu}{\rho}\nabla^2\mathbf{u} + \frac{\lambda + \mu}{\rho}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) + \mathbf{g} \tag{1.3.7}$$

which can be shown to be a wave equation for seismic waves (and can be decomposed into shear waves and compressional waves!). When all the body forces balance the displacements so that there is no time dependence, the equation is said to be the equilibrium condition that

$$\frac{\partial\sigma_{ij}}{\partial x_j} + f_i = 0 \tag{1.3.8}$$

Rotating frames and fictitious forces All of the above equations are derived in an *inertial frame* that is undergoing no accelerations. Although our laboratory frame is actually on a rotating planet (and therefore in an accelerating frame), these equations are usually adequate for most solid-earth problems. The principal exceptions are for flows of low viscosity fluids over large regions of the surface (e.g. ocean, atmosphere and core dynamics). For the most part, however, these fluids are moving only slowly relative to the rate of rotation of the earth and therefore it is convenient to transform the inertial equations into equations relative to a frame rotating with the earth. As it turns out, only the momentum equation is actually effected by this transformation (because spatial derivatives are instantaneous in time and material derivatives are invariant to rotation, e.g. see [1] for a derivation). It can be shown that in a frame rotating at a constant angular velocity $\boldsymbol{\Omega}$ (e.g. 1 revolution per day) the Navier-Stokes Equation becomes

$$\frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V} + 2\boldsymbol{\Omega}\times\mathbf{V} = -\frac{1}{\rho}\boldsymbol{\nabla}P + \nu\nabla^2\mathbf{V} + \mathbf{g}' \tag{1.3.9}$$

where $2\boldsymbol{\Omega}\times\mathbf{V}$ is the *Coriolis Force* and $\mathbf{g}'$ is gravity minus the *centripetal acceleration* (which is about 1/300th of gravity and is negligible) [1]. Both of these forces are "fictitious forces" that arise from the accelerating frame. Nevertheless at least coriolis force has far from fictitious consequences.

1.4 Scaling and "dimensional analysis"

Every term in each of the above equations reflects a physical process and each of these processes have inherent length scales and time scales. If we had to solve for everything, for every problem we would quickly end up in an intractable mess with no hope of salvation. Fortunately, not every process is important in every problem and through judicious use of approximations and *scaling*, we can usually tailor the equations to our problem. The most important tool for determining the relative magnitude of various terms is through *non-dimensional scaling* (often called by the overly fancy name *dimensional analysis*). There are actually three principal purposes to making the equations dimensionless.

1. reduce the number of true parameters.

2. Understand the relative magnitudes of the various processes.

3. Make the equations more tractable for numerical solution (all variables are of order 1).

The mechanics of scaling are straightforward but are best demonstrated with specific examples. Here we will do two basic examples to get the flavor of the exercise. More examples can be found in Chapter 2.

Example 1: Heat flow and the Peclet number The first problem will demonstrate the basic techniques of scaling on a simple two-process problem. In the absence of any heat sources, and assuming constant material properties, the simplest 1-D equation for heat flow is

$$\frac{\partial T}{\partial t} + W\frac{\partial T}{\partial z} = \kappa\frac{\partial^2 T}{\partial z^2} \tag{1.4.1}$$

This equation includes two processes, advection of heat at velocity W, and diffusion of heat with thermal diffusivity κ. As an example problem, this equation could be used to solve for the temperature distribution directly beneath an upwelling mantle plume or ridge (see Figure 1.1a). While it may appear that there are at least two free parameters (W and κ as well as some temperatures), there is in fact only one parameter and it is independent of temperature.

To show this we begin by replacing the dimensional variables with dimensionless ones. The choice of scaling values is a bit of an art. Examination of Figure 1.1a shows that for the case of an upwelling through a thermal layer of depth d, which has constant temperatures T_0 at $z = 0$ and T_1 at $z = d$ and has a characteristic velocity W_0, the sensible scaling is

$$
\begin{aligned}
z &= dz' \\
t &= \frac{d}{W_0}t' \\
W &= W_0 W' \\
T &= T_0 + (T_1 - T_0)T'
\end{aligned}
\tag{1.4.2}
$$

8

where the primes denote dimensionless variables. Formally, scaling is just a transformation between dimensional variables such as $T(z,t)$ to dimensionless ones such as $T'(z',t')$ by use of the chain rule. For example

$$\frac{\partial T}{\partial t} = \frac{dT}{dT'}\frac{\partial T'}{\partial t'}\frac{dt'}{dt} = \frac{\Delta T W_0}{d}\frac{\partial T'}{\partial t'} \tag{1.4.3}$$

where dT/dT' and dt'/dt are calculated from the linear scaling relationships in Eq. (1.4.2) and produce the cluster of parameters $\Delta T W_0/d$ (and $\Delta T = T_1 - T_0$). Similar substitions of Eq. (1.4.2) into (1.4.1) gives

$$\frac{\Delta T W_0}{d}\left[\frac{\partial T'}{\partial t'} + W'\frac{\partial T'}{\partial z'}\right] = \frac{\kappa \Delta T}{d^2}\frac{\partial^2 T'}{\partial z'^2} \tag{1.4.4}$$

Multiplying both sides by $d/(\Delta T W_0)$ and dropping the primes yields (in 1-D where $\mathbf{V} = W_0\mathbf{k}$)

$$\frac{\partial T}{\partial t} + \frac{\partial T}{\partial z} = \frac{1}{\mathrm{Pe}}\frac{\partial^2 T}{\partial z^2} \tag{1.4.5}$$

where

$$\mathrm{Pe} = \frac{W_0 d}{\kappa} \tag{1.4.6}$$

is the *Peclet number* which controls the relative strength of advection to diffusion. If Pe is large, advection dominates and the last term is negligible[2]. If Pe is small, diffusion dominates. However there is only one parameter that controls all solutions. Figure 1.1 shows the analytic steady state solution to (1.4.5) with dimensionless boundary conditions $T(0) = 1$, $T(1) = 0$ and a range of Pe.

Another, more physical way to derive the Peclet number is to consider the time it takes each process to affect the entire layer. The time it takes to advect across the layer at speed W_0 is $t_{adv} = d/W_0$ while the time it takes for heat to diffuse a distance d is $t_{diff} = d^2/\kappa$. Thus the Peclet number is simply the ratio of the diffusion time to the advection time, i.e. $\mathrm{Pe} = t_{diff}/t_{adv}$ (compare also to Eq. (1.4.4)). This is the characteristic property of all the zillions of dimensionless numbers that crop up in continuum mechanics. They are always simply ratios of times or forces generated by any two terms. Thus any two processes will generate a dimensionless number (although if the processes are coupled the overall number of numbers may be less). Based on the sizes of these dimensionless numbers, further approximations are made to simplify the equations.

Example 2: Scaling the quick and dirty way... instant dimensionless numbers
This example will demonstrate the maxim "two processes = 1 number" by demonstrating some quick and dirty scaling arguments for understanding the momentum equation. Consider the dimensional Navier stokes equation for a rotating frame

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V} + 2\boldsymbol{\Omega}\times\mathbf{V} = \nu\nabla^2\mathbf{V} - \frac{1}{\rho}\boldsymbol{\nabla}P + \mathbf{g}' \tag{1.4.7}$$
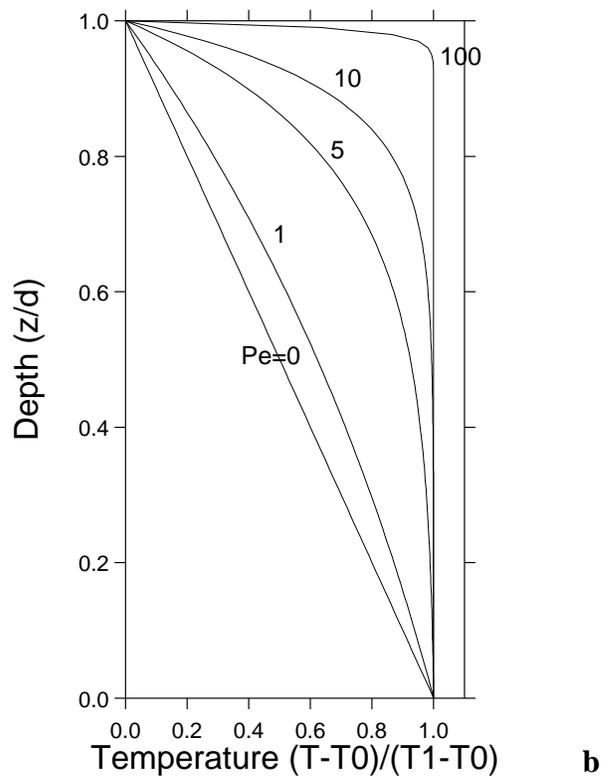

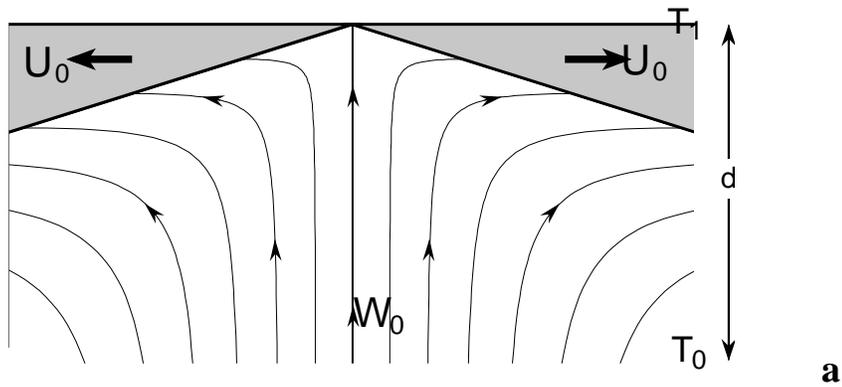
[2]except in narrow boundary layers

Figure 1.1: (a) Cornerflow solution for solid flow beneath a ridge. Layer depth is d, Upwelling rate on axis is W_0. (b) Analytic solution to the simplest steady state-advection diffusion problem. The solution is $T(z) = \left(e^{\mathrm{Pe}z} - e^{\mathrm{Pe}}\right) / \left(1 - e^{\mathrm{Pe}}\right)$. This problem is a good estimate for the thermal structure directly on axis. Note that the Peclet number controls the width of the thermal boundary layers which are of order $l = 1/\mathrm{Pe}$.

which has at least 4 independent processes, advection of momentum, Coriolis force, viscous diffusion and pressure terms. The first trick to Q&D scaling is to simply replace all the derivatives with fractions that have the same units and scales. For example the non-linear advection term $(\mathbf{V} \cdot \boldsymbol{\nabla})\mathbf{V}$ is of the same units and scale as the fraction U_0^2/L where U_0 is a characteristic velocity and L is a characteristic length scale. If we again scale time to the advection time $t_{adv} = L/U_0$ then we can approximate (1.4.7) as

$$\frac{U_0^2}{L} + \frac{U_0^2}{L} + 2\Omega U_0 = \nu \frac{U_0}{L^2} - O(g) \qquad (1.4.8)$$

where all the pressure and body force terms have been lumped into one term of order the body forces. Since all of these terms have units of acceleration (length/time^2) the ratio of any two of the processes is a dimensionless number with a fancy name. It is clear that the first two terms the local acceleration and the advected acceleration are of the same magnitude and we will call the first two terms the *inertial terms* and treat them together. To generate any of the important dimensionless numbers we just consider the relative magnitude of any two terms. For this problem there are three important numbers.

The Reynolds number is the ratio of inertial accelerations to viscous forces i.e.

$$\mathrm{Re} = \frac{U_0 L}{\nu} \qquad (1.4.9)$$

Comparison to Eqs. (1.4.5)–(1.4.6) shows that the Reynolds number and the Peclet number fill the same role. Thus the Reynolds number controls the size of viscous boundary layers. Large Re means turbulent flow with thin viscous boundary layers. Small Re flow are strongly viscous.

The Rossby number is the ratio of inertial accelerations to coriolis accelerations i.e.

$$\mathrm{Ro} = \frac{U_0}{2\Omega L} \qquad (1.4.10)$$

If $\mathrm{Ro} \to 0$, the system is effectively in solid body rotation. For $\mathrm{Ro} > 1$ rotation can be neglected relative to *inertial forces* (e.g. laboratory flume experiments). Many interesting problems happen for Rossby numbers of order 1.

The Eckman number is the square root of the ratio of viscous forces to coriolis forces

$$\mathrm{Ek} = \left[\frac{\nu}{2\Omega L^2}\right]^{1/2} \qquad (1.4.11)$$

The Eckman number controls the size of boundary layers in rotational problems. For large Ek viscous forces dominate and coriolis terms can be neglected.

Table 1.1 gives some representative values of scales and dimensionless numbers for the earth's mantle, the Gulf Stream, and your bathtub. By inspecting the

Table 1.1: Some scales and scaling for three fluid problems.

	The mantle	The Gulf Stream	Your Bathtub
U_0	3 cm/yr (10^{-9} ms^{-1})	1 ms^{-1}	.01 ms^{-1}
L	3000 km	100 km	1 m
ν	10^{18} m^{2}s^{-1}	10^{-6} m^{2}s^{-1}	10^{-6} m^{2}s^{-1} (10^{-2} m^{2}s^{-1} corn syrup)
Ω	2π/day (7.3×10^{-5} s^{-1})	7.3×10^{-5} s^{-1}	7.3×10^{-5} s^{-1}
Re	3×10^{-21}	10^{11}	10^4 (1 corn syrup)
Ro	2×10^{-12}	.07	70
Ek^2	8×10^8	7×10^{-13}	7×10^{-3} (70 for cs)

relative magnitudes of the different processes, this quick scaling suggests that for mantle convection we can neglect inertial and rotational forces. For the oceans we can neglect (with some care) viscous forces, and for our bathtub we can neglect coriolis forces (thus destroying the direction of bathtub draining myth).

Caveats: Small parameters This approach is good for a quick back of the envelope estimate of the relative magnitude of different terms. Usually when one term is much smaller than another it can just be thrown out (at least as a first guess). Some caution should be exerted however when the small value multiplies the terms with the highest derivatives (e.g. $(1/\text{Pe})\nabla^2 T$ in example 1). As demonstrated in figure 1.1b, a large Peclet number does not necessarily imply that diffusion can be neglected, rather it means that it only becomes important in narrow boundary layers. Without these boundary layers, however, some problems are poorly posed. A more drastic example from magma migration will be shown in class.

1.5   Summary

Beginning with conservation of anything for a fixed volume, we have come up with a large number of equations that govern the physics of just about every continuum problem we can think of. Using scaling, we have also shown how to simplify (?) these equations and how to get the first vague understanding of the relative importance of different processes for different problems. Now it's time to start solving these equations. Fortunately there are really only three basic types of equations that come out of this analysis, Ordinary Differential Equations which depend only on time, Time dependent Partial differential equations (space and time) and boundary value problems (just space). The following sections will show how to deal with each of these basic types in turn.

Bibliography

[1] M. Ghil and S. Childress. Topics in geophysical fluid dynamics : atmospheric dynamics, dynamo theory, and climate dynamics, vol. 60 of Applied mathematical sciences, Springer-Verlag, New York, 1987.