4 Problem set #4: Fun with diffusion

Today's thrill packed exercise will be to deal with diffusion and advection-diffusion in one dimension. All exercises here will be in Matlab and will build upon example codes in `probset4/matlab/`. I will provide two fully worked out codes for crank-nicolson diffusion of a gaussian initial condition with dirichlet boundary conditions (`Diffusion/diffusion_cn.m`) and a semi-lagrangian advection-radioactive decay program (`Advection/advect_semi_lag.m`). See the README files in all directories.

Choose **one** of the following problems (extra credit if you do them both).

1. **Fun with boundary conditions**: This problem modifies the pure-diffusion example code to investigate the behaviour of a problem with time dependent, oscillating boundary conditions. This is a useful model problem for understanding how temperature profiles in the ground might evolve due to either daily or seasonal temperature fluctuations.

   Consider the dimensional diffusion problem

   $$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$

   for a layer of depth L and with initial conditions $T(x, 0) = 0$ and boundary conditions

   $$T(0, t) = T_{max} \sin(2\pi t/P) \qquad \frac{\partial T}{\partial x}(L, t) = 0 \tag{4.1}$$

   I.e. consider an oscillating surface temperature where P is the period of oscillation of the surface boundary condition, and T_{max} is the maximum deviation from the mean temperature. The other boundary condition is a Neumann, no-flux condition.

   (a) **Scaling** What is the appropriate scaling to transform Eq. (4.1) into

   $$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \tag{4.2}$$

   What are the boundary conditions under this new scaling and what is the *dimensionless* period of oscillation P'?

   (b) **More Scaling** If the layer thickness L is 10m, and the thermal diffusivity of the rock is $\sim 10^{-6}$ m^{2}s^{-1} what is the approximate diffusion time scale for heat to cross the layer? How far do you expect that a daily temperature oscillation $P = 1$ day to penetrate (hint, what is P' for this period?). What about $P = 1$ year?

(c) Test your intuition by solving the dimensionless equations numerically. Modify the Crank-Nicolson code `diffusion_cn.m` to implement the new boundary conditions. (Hint: you might find it easier to understand the results of the run if you change the inputs to input the dimensionless period P, the number of periods you want to run the program `n_perod`, and the number of plots per period `n_plot`. Using these new parameters it will be easier to calculate `t_max` and `t_save`.

(d) Now consider the thermal evolution of a layer 10 meters thick with thermal diffusivity $\kappa = 10^{-6}$ m^{2}s^{-1}. Using your scaling from the previous section solve for the thermal profiles as a function of time given a daily oscillation ($P = 1$day), an intermediate monthly oscillation ($P = 30$ days) and a yearly oscillation. (Beware the bottom boundary here, you might need a deeper layer). In all cases, what is approximately the maximum temperature at a depth $x = P$?

2. **Advection-Diffusion–making it move**: The general dimensional advection-diffusion equation in 1-D is

$$\frac{\partial T}{\partial t} + \frac{\partial vT}{\partial x} = \frac{\partial}{\partial x}\kappa\frac{\partial T}{\partial x} \tag{4.3}$$

where T is the temperature v is a transport velocity, and κ is the thermal diffusivity (this equation already assumes that ρc_P is a constant).

(a) Assume that the velocity and diffusivity are constant ($v = v_0$, $\kappa =$const). Show that by non-dimensionalizing by the *diffusion time*, Eq. (4.3) can be written

$$\frac{\partial T'}{\partial t'} + \mathrm{Pe}\frac{\partial T'}{\partial x'} = \frac{\partial^2 T'}{\partial x'^2} \tag{4.4}$$

where primes denote dimensionless variables and $\mathrm{Pe} = v_0 d/\kappa$ is the Peclet number and d is some characteristic lengthscale.

(b) Show that if we make the transformation $T(t, x) = T(\tau, \zeta)$ where

$$\begin{aligned} \tau &= t' \\ \zeta &= x' - \mathrm{Pe}t' \end{aligned} \tag{4.5}$$

that (4.4) can be rewritten as

$$\frac{\partial T'}{\partial \tau} = \frac{\partial^2 T'}{\partial \zeta^2} \tag{4.6}$$

i.e. we have apparently removed the advective term. What is the physical meaning of this transformation? (FYI it's called a Galilean Transformation).

(c) Confirm that for an infinite 1-D medium

$$T' = 1 + \frac{A}{\sqrt{1 + 4\tau/\sigma^2}} \exp\left[\frac{-\zeta^2}{\sigma^2 + 4\tau}\right] \tag{4.7}$$

is a solution of (4.6) for a gaussian initial condition of amplitude A (over a background temperature of 1) and half-width σ and therefore

$$T' = 1 + \frac{A}{\sqrt{1 + 4t'/\sigma^2}} \exp\left[\frac{-(x' - x_0 - \mathrm{Pe}t')^2}{\sigma^2 + 4t'}\right] \tag{4.8}$$

is a general solution for (4.4) when the gaussian peak is initially at position x_0 (you don't have to redo the whole calculation just show that these sorts of solutions are invariant to galilean transformations and therefore translations).

(d) Derive (or simply write down) the finite difference updating schemes for Eq. (4.4) and the following approaches

i. "All-in-one" Crank-Nicholson scheme with explicit advection terms (i.e. $\partial T/\partial t = (\mathcal{L}T^{n+1} + \mathcal{L}T^n)/2$ where $\mathcal{L}$ is the differential operator for the spatial terms including both advection and diffusion).

ii. Operator splitting with semi-lagrangian advection and Crank-Nicolson diffusion.

iii. All-in-one combined Semi-Lagrangian-Crank-Nicolson scheme (i.e. $DT/Dt = \partial^2 T/\partial x^2$)

For convenience, you can drop the primes.

(e) For a fixed grid spacing Δx, what are the two intrinsic time scales for advection-diffusion problems? Show how you would choose an appropriate time-step Δt given Pe and Δx.

(f) Choose one (or more if you're ambitious) of the above schemes and write a program to solve this problem (feel free to modify the example codes).

(g) Use your solver to solve for a gaussian initial condition at $t = 0$ with initial peak at $x_0 = 15$ (see the useful utility program is `diffuse_gaussian.m`) and dirichlet boundary conditions $T(0, t) = T(50, t) = 1$. Solve using $\mathrm{Pe} = 0, 5, 10, 50, 100, 500$ for a gaussian that initially has a peak at $x_0 = 15$ but moves a **fixed** distance of 20 during the run (careful for $\mathrm{Pe} = 0$). Use the analytic solution to calculate errors and discuss the accuracy of your scheme as a function of numerical parameters and the Peclet number.