Uncertainties in Estimates of Reynolds Stress and TKE Production Rate Using the ADCP Variance Method

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ABSTRACT

The use of acoustic Doppler current profilers (ADCPs) to measure turbulent parameters via the variance method involves uncertainties due to instrument noise and flow-related errors in measurement. For weak flows, the uncertainty in Reynolds stress measurements arises mainly from instrument noise and is proportional to the square of the velocity standard deviation, while the uncertainty in the corresponding estimates of the rate of production of turbulent kinetic energy (TKE) is proportional to the cube of the velocity standard deviation. For stronger flows, the principal determining parameter is the number of individual independent velocity measurements over which the variance is calculated. These results are validated by detailed analyses of two datasets from an RD Instruments 1.2-MHz Workhorse ADCP, using a ping rate of 2 Hz with ensemble averaging at 0.5 Hz, and a ping rate of 10 Hz with ensemble averaging at 1 Hz, respectively. While increasing ping rate generally reduces the effects of instrument noise, it will not alleviate the influence of flow-related noise once the sampling interval is less than the autocovariance time scale of the turbulence. Using the fast ping rate, the uncertainty in the Reynolds stress due to instrument noise is reduced by a factor of more than 3 to 0.02 Pa; in higher energy environments there is a reduction in the uncertainty of about 30%. The observational and theoretical estimates for the reduction in the uncertainty using the fast ping rate are in good agreement.

1. Introduction

The measurement of turbulent parameters in tidal flows presents oceanographers with a difficult problem, since it is necessary to obtain rapid measurements of velocity at small spatial scale in an environment in which large stresses operate and where, away from the bottom boundary, there is no fixed reference for velocity measurement. In recent years, the development of free-fall profilers, which measure velocity shear at scales down to a few millimeters, has made possible high quality measurements of turbulent dissipation (e.g., Dewey et al. 1987; Simpson et al. 1996). The method is, however, very labor intensive and therefore limited to short observational periods. A recent alternative development in turbulence studies is a method of determining the Reynolds stresses (\(\overline{-u'w'}\), etc.) using a high-frequency acoustic Doppler current profiler (ADCP). This approach enables us to obtain much longer time series of turbulent parameters without the labor and ship time costs of the free-fall profilers.

The estimation of Reynolds stresses and the rate of production of turbulent kinetic energy (TKE) from the variances of the along-beam velocities of an ADCP has its origins in radar meteorology [see Rottger and Larsen (1990) for an overview]. It was developed for use in a marine environment by Tropea (1983) using a one-dimensional laser Doppler anemometer and applied to profiling of stresses in a shelf sea environment by Lohmann et al. (1990) using a pulse-to-pulse coherent acoustic profiler. The method has been further developed in differing shallow water environments using high-frequency ADCPs (e.g., Stacey et al. 1999; Lu and Lueck 1999; Rippeth et al. 2002).

While efforts have been made in previous studies to quantify the uncertainties for a particular dataset, a question that remains to be addressed is the general relationship between the uncertainties in the velocity measurements and the estimates of Reynolds stress and TKE production rate. An understanding of how the uncertainties in these estimates differ with different ADCP configurations is essential if we are to optimize future measurements. Instrument manufacturers are introducing fast-pinging modes of sampling, which decrease the standard deviation of velocity measurements. In this article we consider how the lower uncertainty in the velocity measurements can best be exploited to reduce the uncertainties in the turbulence estimates. A theoretical method of quantifying the uncertainties is outlined here; this theory can be applied to the configuration for future observations in order to obtain the best possible quality of measurements.

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2. Data collection

Two datasets were collected, one in the Menai Strait, north Wales, during June–July 2000 and the other in the York River estuary, Virginia, during March 2002. In each case, a 1.2-MHz RD Instruments (RDI) Workhorse ADCP was moored on the bed and used to record along-beam velocities with a depth cell size of 0.5 m over a period of approximately 72 h.

The Menai Strait is a narrow, tidally energetic channel approximately 20 km in length, which separates the island of Anglesey from the mainland of Great Britain. The depth varies along its length, with a maximum of about 22 m; at the site chosen for the observations the mean depth was approximately 15 m with a tidal range at springs of about 5 m. The strong tidal currents reach speeds of up to 2.5 m s⁻¹ at springs in the shallower regions. There is a tidal asymmetry in the flow that produces a stronger acceleration in the flow during the transition from flood to ebb resulting in a residual flow in the ebb direction (Simpson et al. 1971). The strong tidal currents and the relatively small freshwater input ensure that the water column is almost always well mixed; the along-channel density gradient is also weak (Campbell et al. 1998). In the Menai Strait deployment, RDI’s standard mode 1 was used, with a ping rate of 2 Hz; velocities were averaged by the instrument into 2-s ensembles of four pings before recording. This dataset will be referred to throughout as mode 1.

The York River estuary is partially mixed estuary that discharges into Chesapeake Bay. It is tidally energetic, with surface currents of up to about 1 m s⁻¹ at springs, but the tidal range is much smaller than that of the Menai Strait, with a maximum of about 1 m. The site chosen for the observations the mean depth was approximately 6.5 m. The estuary is periodically stratified, with maximum stratification occurring during the ebb at neaps; during the spring tide, the stratification is eroded, and at times the water column is completely vertically mixed (Sharples et al. 1994). During the period used for this analysis, which was close to springs, only slight stratification was observed during the ebb, with complete vertical mixing on the flood. In the York River deployment, RDI’s fast-pinging mode 12 was used, with a ping rate of 10 Hz; velocities were averaged and recorded every second. This dataset will be referred to as mode 12.

Finally, further estimates of noise levels were made from along-beam velocities measured in an almost stationary water column in Vivian quarry near Llanberis, north Wales, in an attempt to quantify the instrument noise. The instrument was deployed for short periods facing downward through the water column in different configurations, each time using 0.5-m-depth cells, and the along-beam velocities recorded as for the two datasets. These data will be referred to as the quarry data.

3. Data analysis

The data were analyzed using the “variance” method, which is described in some detail by Lu and Lueck (1999). Using a 10-min averaging period, each along-beam velocity measurement \( b_i \) was split into a mean and a fluctuating part \( \bar{b}_i \) and \( b'_i \), respectively:

\[
\bar{b}_i = \bar{b}_i + b'_i. \tag{1}
\]

Using the two pairs of opposite beams, and assuming homogeneity of the statistical properties of the flow across the spread of the beams, the components of the Reynolds stress \(-u'w'\) and \(-v'w'\) were calculated from the variances of the along-beam velocities according to

\[
\frac{\tau_u}{\rho} = -u'w' = \frac{\bar{b}_1^2 - b_1'^2}{4 \sin \theta \cos \theta} \quad \text{and} \quad \frac{\tau_v}{\rho} = -v'w' = \frac{\bar{b}_2^2 - b_2'^2}{4 \sin \theta \cos \theta}, \tag{2}
\]

where \( \theta \) is the angle each beam of the ADCP makes with the vertical; in the case of the RDI Workhorse ADCP, this angle is 20°.

### a. Analysis of uncertainties in Reynolds stress estimates

If we choose the \( x \) coordinate to be in the main direction of flow, the variance of the principal Reynolds stress components is given by (Stacey et al. 1999)

\[
\text{var}(\bar{u}'w') = \sigma_{\bar{u}'w'}^2 = \frac{1}{16 \sin^2 \theta \cos^2 \theta} \left[ \text{var}(\bar{b}_1^2) - \text{cov}(\bar{b}_1^2, b_1'^2) \right]. \tag{3}
\]

The magnitude of the uncertainty can be calculated directly from the data if the components are rewritten as follows, using \( M \) ensembles to calculate the variances:

\[
\sigma_{\bar{u}'w'}^2 = \frac{1}{16 \sin^2 \theta \cos^2 \theta} \times \text{var} \left[ \frac{1}{M} \sum_{i=1}^{M} \bar{b}_1^2(m) - \frac{1}{M} \sum_{i=1}^{M} b_1'^2(m) \right]. \tag{4}
\]

This can be expanded to give (see appendix A for details)

\[
\sigma_{\bar{u}'w'}^2 = \frac{\sum_{i=1}^{M} \sum_{m=1}^{M} \text{var}(\bar{b}_1^2(m)) + 2 \sum_{i=1}^{M-1} \sum_{n=m+1}^{M} \text{cov}(\bar{b}_1^2(m), b_1'^2(n)) - 2 \sum_{m=1}^{M} \sum_{n=1}^{M} \text{cov}(\bar{b}_1^2(m), \bar{b}_1^2(n))}{16M^2 \sin^2 \theta \cos^2 \theta}. \tag{5}
\]
In the two datasets considered here, the last term on the right-hand side of Eq. (5), representing the sum of the covariance of the squares of the fluctuations in opposite beams, was found to be an order of magnitude smaller than the first term, which represents the sum of the variances of the squares of the fluctuations in each of the two beams; hence, the last term will be neglected. The second term represents the correlation between the square of each velocity fluctuation and subsequent measurements. This correlation can be expressed in terms of the normalized autocovariance function \( \rho \) given by

\[
\rho(m, n) = \frac{\text{cov}[b_i^2(m), b_i^2(n)]}{\sqrt{\text{var}[b_i^2(m)] \text{var}[b_i^2(n)]}}.
\]

In stationary flow, the correlation should be a constant for the period of the \( M \) ensembles used in the variance calculations; hence, \( \text{var}[b_i^2(m)] = \text{var}[b_i^2(n)] \) and \( \rho(m, n) = \rho(m, m + n - 1) \). An upper limit (\( n = K \)) can be defined, above which the correlation terms become negligibly small. Here, \( K \) is small compared to \( M \) since the correlation effects of velocity fluctuations only extend over periods of the order of 20 s or less (Stacey et al. 1999; Lu and Lueck 1999), so the sum of the covariances can be simplified (Heathershaw and Simpson 1978):

\[
\sum_{m=1}^{M-1} \sum_{n=m+1}^{M} \text{cov}[b_i^2(m), b_i^2(n)] \\
\approx M \sum_{n=2}^{K} \text{cov}[b_i^2(1), b_i^2(n)].
\]

The sum of the covariances for lags greater than zero is then given by

\[
\sum_{n=2}^{K} \rho(1, n) \approx \frac{K-1}{M} \sum_{m=1}^{M} \text{cov}[b_i^2(m), b_i^2(n)].
\]

For large \( M \), \( (M - 1)/M \rightarrow 1 \), so Eq. (5) becomes

\[
\sigma_k^2 = \frac{\sum_{i=1}^{M} \left[ \text{var}[b_i^2(m)] \left[ 1 + 2 \sum_{n=2}^{K} \rho(1, n) \right] \right]}{16M^2 \sin^2 \theta \cos^2 \theta}.
\]

Using the stationarity condition,

\[
\sum_{m=1}^{M} \text{var}[b_i^2(m)] = M \text{var}(b_i^2),
\]

we can write

\[
\sigma_k^2 = \frac{1}{16M \sin^2 \theta \cos^2 \theta} \left[ \sum_{i=1}^{M} \left[ \text{var}(b_i^2) \left[ 1 + 2 \sum_{n=2}^{K} \rho(1, n) \right] \right] \right]
\]

\[
= \frac{\gamma_k \text{var}(b_i^2) + \text{var}(b_i^2)}{16M \sin^2 \theta \cos^2 \theta},
\]

where

\[
\gamma_k = 1 + 2 \sum_{n=2}^{K} \rho(1, n).
\]

The factor \( \gamma_k \) is therefore a correction factor to account for the nonindependence of consecutive measurements of velocity fluctuations. When adjacent velocity fluctuations are independent, the sum of the autocorrelations is zero, and \( \gamma_k = 1 \). Conversely, as adjacent velocity fluctuations become increasingly covariant, \( \gamma_k \) increases, with a consequent increase in \( \sigma_k^2 \), which offsets the decrease in \( \sigma_k \) obtained through the increase in \( M \).

It is assumed here that the autocovariance time scales of the squares of the fluctuations, and hence the factor \( \gamma_k \), are the same for two opposite beams. In the case of the present datasets, this was found to be a reasonable assumption: Fig. 1 shows the estimated values of \( \gamma_k \) for all four beams for the flood and ebb phases of the tide for the mode 1 and mode 12 datasets. The factor \( \gamma_k \) is slightly higher when the horizontal component of the along-beam velocity is in the direction of the mean along-channel flow.

In order to analyze the noise characteristics of the Reynolds stress estimates, a relationship between \( \text{var}(b_i^2) \) and the along-beam velocity measurements is required. First, we obtain an expression for \( \text{var}(b_i^2) \) in terms of the second and fourth moments of the measured along-beam velocities:

\[
\text{var}(b_i^2) = \frac{1}{M} \sum_{i=1}^{M} (b_i^2 - \bar{b}_i^2)^2 = \mu_4 - \mu_2^2.
\]

If the distribution is Gaussian, then \( \mu_4 = 3\mu_2^2 \). Analysis
of the two datasets shows that the mean value of $\mu_u/\mu^2_w$ is 3.02 for mode 1 and 3.1 for mode 12. So we may treat the distribution as Gaussian and write

$$\text{var}(b_z^2) = 2(b_z^2)^2.$$  \hspace{1cm} (10)

Hence,

$$\sigma^2_g = \frac{\gamma_g(b_z^2)^2 - (b_z^2)^2}{2M \sin^2\theta}.$$  \hspace{1cm} (11)

The calculated along-beam variance comprises the true variance and the instrument noise (Stacey et al. 1999):

$$\sigma^2_g = \frac{\gamma_g(\sigma_w^2 + \text{cov}(x, N))}{2M \sin^2\theta}.$$  \hspace{1cm} (13)

As $x_i \to 0$ and $\bar{x}^2 \to 0$, $\sigma^2_g \to (\gamma_g \sigma_w^2/M \sin^2\theta)$. If the variance is due to noise alone, there is no correlation between one measurement and the next and $\gamma_g = 1$, giving

$$\sigma^2_g = \frac{\sigma_w^2}{M \sin^2\theta}.$$  \hspace{1cm} (14)

This sets the minimum measurable value of $\tau$, which is dependent solely on the instrument noise.

The implications of Eq. (13) are as follows. For low-flow situations, when the value of $\bar{x}^2$ is small compared to $\sigma^2_g$, $\sigma^2_g$ can be reduced by lowering the instrument noise. Further reductions in $\sigma^2_g$ can be obtained if the number of ensembles ($M$) used to calculate the variance is increased. For stronger flows, when $\bar{x}^2$ dominates over $\sigma^2_g$, the only way in which $\sigma^2_g$ can be reduced appreciably is to increase the value of $M$. An additional implication of Eq. (13) is that a beam angle of 30° instead of 20° would reduce the standard deviation of the Reynolds stress estimates by 26%.

b. Uncertainty in shear estimates

The uncertainty in the estimate of the shear is given by

$$\text{var} \left( \frac{\partial u}{\partial z} \right) = \frac{1}{(\Delta z)^2} \text{var}(u_{n+1} - u_{n-1}),$$  \hspace{1cm} (15)

where $u$ is the horizontal velocity in the $x$ direction. The variance in the horizontal velocity, $\text{var}(u)$, is calculated using the instantaneous horizontal velocity obtained from two opposite beams, assuming the vertical velocity is the same in both beams:

$$u = \frac{(b_z - b_z)}{2 \sin\theta}.$$  \hspace{1cm} (16)

Hence when $M$ ensembles are used to obtain the final shear estimate, again incorporating a factor $\gamma_s$ to account for the correlation between one shear estimate and the next, we have

$$\text{var} \left( \frac{\partial u}{\partial z} \right) = \frac{\gamma_s \{ \text{var}(u_{n+1} - u_{n-1}) + \text{cov}(x, N) \}}{4M(\Delta z)^2 \sin^2\theta}.$$  \hspace{1cm} (17)

The assumption that the vertical velocity is the same in both beams will increase the calculated variance if the two vertical velocities are not the same; hence, Eq. (17) will tend to overestimate the variance of the shear. It is readily seen from this equation that the uncertainty in the shear can be reduced by increasing the number of ensembles to be averaged. It can also be reduced by increasing the depth cell size, at the expense of the vertical resolution.

c. Analysis of uncertainties in TKE production rate estimates

The error in the estimates of rate of shear production can be determined using the formula for the variance of a product. For two independent variables, this is (Goodman 1960)

$$\text{var}(xy) = \pi^2 \text{var}(y) + \pi^2 \text{var}(x) + \text{var}(x) \text{var}(y).$$

So in terms of our present notation, where we are interested in the variance of $P = -\bar{u}' \bar{w}' \left( \frac{\partial u}{\partial z} \right)$,

$$\sigma^2_P = \bar{u}' \bar{w}' \sigma^2_u + \left( \frac{\partial u}{\partial z} \right) \sigma^2_g + \sigma_u^2 \sigma_g^2.$$  \hspace{1cm} (18)

For the low-flow case, $\bar{u}' \bar{w}' \to 0$ and $\left( \frac{\partial u}{\partial z} \right) \to 0$, so it is expected that the last term will dominate. The other two terms are expected to dominate at times of higher flow. In the next section, the effect of the other two terms will be examined and strategies for reducing the uncertainties will be discussed. A further discussion
of the justification for assuming independence, and the
derivation of the formula for the variance of a product,
can be found in appendix B.

4. Results

Figure 2 shows the standard deviation of the Reynolds
stresses calculated from the observations using Eq. (11)
plotted against the associated Reynolds stress. The
factor \(\gamma_R\) for the Reynolds stresses was estimated as having
a median value of \(\sim 1.6\) for mode 1 and \(\sim 1.7\) for mode
12 (Fig. 1).

If a linear regression is performed on the data shown
in Fig. 2, we obtain the following:

mode 1,

\[
\rho \sigma_R = 0.151 |u'| w^2 + 0.074 \quad (r^2 = 0.93); \quad \text{and}
\]

mode 12,

\[
\rho \sigma_R = 0.104 |u'| w^2 + 0.018 \quad (r^2 = 0.95).
\]

The intercept represents the instrument noise, which is
reduced from \(-0.07\) Pa in mode 1 to less than \(-0.02\)
Pa in mode 12. By referring to Eq. (14), it is apparent
that this reduction is due to a combination of increasing
the number of pings per ensemble, which reduces \(\sigma_N\)
and hence \(\sigma_R\), and increasing \(M\), which also reduces \(\sigma_R\).

At higher stress levels, there are two components in the
uncertainty in the Reynolds stress. These are the in-
strument noise component and the flow-related com-
ponent due to turbulent motions. The gradient in the
two equations above represents the increase in the flow-
related component of the uncertainty and is affected
only by the number of ensembles used to estimate the
stress. The reduction in the gradient from 0.151 to 0.104
indicates the reduction in the flow-related component
of the uncertainty due to using 600 ensembles instead
of 300. The reduction is consistent with the theory, since
if twice as many ensembles are used to calculate the
Reynolds stress, the standard deviation due to the tur-
bulent fluctuations should be reduced by a factor of
\(\sqrt{2} \approx 1.41\). Here, the reduction is by a factor of \(\sim 1.45\).

An estimate of the total reduction in the uncertainty must
also take into account the lower intercept when using
mode 12; the benefits of using mode 12 are therefore
greater in low stress regimes than in very strong tur-
bulence.

The along-beam velocity standard deviation \((\sigma_N)\)
from the trials at the quarry were estimated as 0.0217
m s\(^{-1}\) for mode 1 (using a four-ping mean as in the
observations) and 0.0184 m s\(^{-1}\) for 1-s averages at 10
Hz in mode 12. Using the data from the two deploy-
ments, we get values of 0.018 m s\(^{-1}\) for mode 1 and
0.014 m s\(^{-1}\) for mode 12. These are slightly higher than
RDI’s values of 0.0168 m s\(^{-1}\) for mode 1 and 0.0106
m s\(^{-1}\) for mode 12.

For still water, both the along-beam velocity and the
variance due to turbulent fluctuations are zero, so we
can determine the variance in the Reynolds stress using
Eq. (14). From the quarry results, the standard deviation
of the Reynolds stress due to instrument noise is
\(\rho \sigma_N = 0.0436\) Pa in mode 1 (using \(M = 300\)) and
\(\rho \sigma_N = 0.0219\) Pa in mode 12 (using \(M = 600\)). The larger
value obtained for the minimum standard deviation of
the Reynolds stress for mode 1 using the regression
method is due to the greater scatter of the data points
around the regression line, despite a good fit to the line.
This can be seen clearly in Fig. 2, where the median
value of the standard deviation for near-zero Reynolds
stress is less than 0.06 Pa, consistent with the quarry
value.

The uncertainties in the Reynolds stresses were ob-
tained using the semiempirical method outlined in sec-
tion 3a. In order to check on the accuracy of this method,
further estimates of the uncertainties were made directly
from the data. The Reynolds stresses for depth cell 2
were sorted according to the associated value of the
Reynolds stress in depth cell 1, since a linear regression
of the stress in cell 2 on the associated stress in cell 1
yields a correlation coefficient, \(r^2\), of more than 97% for
both datasets (Figs. 3a and 3b). The data were sorted
into 24 bins, with an equal number of data points in
each. For each bin the standard deviation was calculated
as the rms value of the deviations from the linear trend-
line and plotted against the mean Reynolds stress for
that bin. The results for this are shown in Fig. 4, together
with the semiempirical values obtained from Eq. (11).
Further estimates of the uncertainty at hourly intervals
for a 25-h subset of the mode 12 data were made using
a bootstrap method, following that used by Lu and
Lueck (1999), in which the along-beam velocity fluc-
tuations were resampled to obtain 1000 new series of
600 data points for each 10-min period, which were then
used to obtain 1000 new estimates of the “stress.” The 95% confidence limit obtained using this method is in very close agreement with the value of $2\sigma_S$ obtained using the other methods.

For still water, we have the variance of the shear from Eq. (17):

$$\sigma_S^2 = \frac{\sigma_N^2}{M(\Delta z)^2 \sin^2 \theta}.$$  (19)

For mode 1, the minimum value of $\sigma_S$ from the observational data is $4.19 \times 10^{-3}$ s$^{-1}$. From the quarry noise tests, the standard deviation of the shear due to instrument noise alone is $3.67 \times 10^{-3}$ s$^{-1}$, consistent with the observational values. For mode 12 the results are even closer: $2.20 \times 10^{-3}$ s$^{-1}$ from the quarry and $2.09 \times 10^{-3}$ s$^{-1}$ from the observations.

Figure 5 shows a similar plot to Fig. 2; this time for the standard deviation of the TKE production rate plotted against the production rate. The gradient here is reduced from $-0.194$ to $-0.128$ ($r^2$ values are approximately 0.97 for both), indicating that the percentage error in the TKE production rate estimates has been reduced from over 19% to less than 13%. The reduction factor here is $\sim 1.5$, compared to a factor of $\sqrt{2}$ from theory.

As for the Reynolds stresses, a direct method was used to check the accuracy of the calculated values of the standard deviation of the TKE production rate, which were obtained using the method outlined in section 3. The values of $P$ in depth cell 2 were binned according to the associated value of $P$ in depth cell 3, since again, there is a strong linear correlation between the estimated value of $P$ in adjacent depth cells (Figs. 3c and 3d) with $r^2$ values of 93% for mode 1 and 96% for mode 12. Removing the linear trend, we obtain the standard deviation of $P$, represented by the rms value of the deviations from the line. These results are shown in Fig. 6. The direct estimates of the standard deviations of TKE production rate are in satisfactory agreement with the semiempirical values for both mode 1 and mode 12.

The standard deviations of the TKE production rate due to each of the three terms in Eq. (18) are plotted for both mode 1 and mode 12 in Fig. 7, along with the total standard deviation. About 85% of the variance in the TKE production rate estimates is due to the second term in Eq. (18) $\rho v^2(\partial u/\partial z)^2\sigma_S^2$ at times of high flow, with most of the remaining 15% due to the first term $\rho v^2(\partial^2 u/\partial z^2)^2\sigma_S^2$ and the last term $\rho^2\sigma_\theta^2\sigma_S^2$ making a negative contribution.

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The std devs of the TKE production rate estimates using the two methods are compared for (a) mode 1 and (b) mode 12. In the empirical method, shown by the open circles, the data were sorted into bins containing an equal number of data points according to the value of the production rate in an adjacent depth cell; the mean and std devs were then calculated. In the semiempirical method, shown by crosses, the std devs were calculated from Eqs. (11), (17), and (18).

The estimated values of the standard deviations of the along-beam velocity ($\sigma_{\bar{u}}$), the Reynolds stress ($\sigma_{\overline{\tau}}$), the shear ($\sigma_s$), and the rate of production of TKE ($\sigma_{P_\tau}$) for mode 1 and mode 12. Each value has been estimated both from the observations and from the tests in a stationary water column (the quarry tests) using a 10-min averaging period.

<table>
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<tr>
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<th>Mode 1</th>
<th>Mode 12</th>
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<tbody>
<tr>
<td>$\sigma_{\bar{u}}$ (obs) (m s$^{-1}$)</td>
<td>0.018</td>
<td>0.014</td>
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<tr>
<td>$\sigma_{\bar{u}}$ (test) (m s$^{-1}$)</td>
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<td>0.0184</td>
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</tr>
<tr>
<td>$\rho\sigma_{\overline{\tau}}$ (test) (Pa)</td>
<td>0.0435</td>
<td>0.0219</td>
</tr>
<tr>
<td>$\sigma_s$ (obs) (s$^{-1}$)</td>
<td>$4.19 \times 10^{-3}$</td>
<td>$2.09 \times 10^{-3}$</td>
</tr>
<tr>
<td>$\sigma_s$ (test) (s$^{-1}$)</td>
<td>$3.67 \times 10^{-3}$</td>
<td>$2.20 \times 10^{-3}$</td>
</tr>
<tr>
<td>$\rho\sigma_s$ (obs) (W m$^{-3}$)</td>
<td>$2.12 \times 10^{-4}$</td>
<td>$3.01 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\rho\sigma_s$ (test) (W m$^{-3}$)</td>
<td>$1.60 \times 10^{-4}$</td>
<td>$4.81 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

The instrument noise, $\rho\sigma_s\sigma_{\overline{\tau}}$, shown by the open circles in Fig. 7, has a minimum value of about $2.1 \times 10^{-4}$ W m$^{-3}$ for mode 1 and about $3 \times 10^{-5}$ W m$^{-3}$ for mode 12. Inserting the values from the quarry noise tests into Eq. (14) and (19) give noise levels in TKE production rate $\rho\sigma_s\sigma_{\overline{\tau}}$ of $1.6 \times 10^{-4}$ W m$^{-3}$ and $4.8 \times 10^{-5}$ W m$^{-3}$ for modes 1 and 12, respectively, consistent with the noise levels from the observations.

The estimates of the uncertainties in the Reynolds stress, shear, and TKE production rate are summarized in Table 1. Comparing the estimate of each parameter using the quarry test results and the observational data shows that there is agreement in each case to within a factor of 2 or better. These results indicate that the still water tests are a good indicator of the true uncertainties in Reynolds stress, shear, and TKE production rate estimates.

To illustrate the improvement in the quality of the data obtained using mode 12, Reynolds stress profiles at hourly intervals for a 12-h period for each dataset are shown in Fig. 8. The profiles are those obtained using Eq. (2), with a 10-min averaging period and represent the flow in the along-channel direction. In Figs.
8a and 8b the 10-min profiles are shown for modes 1 and 12, respectively; in Figs. 8c and 8d the profiles have been averaged over an hour (six profiles).

In general, the stress profiles show the expected pattern of increasing magnitude toward the bed, and tending toward zero near the surface. In both datasets, the magnitude of the stresses and the mean flow speed is greater on the ebb.

Error bars (±1 standard deviation) are indicated in each case for the two profiles at the extremes of the tidal cycle; these are the maximum uncertainties, since, except close to slack water, the uncertainty is mainly due to the element that is proportional to the stress. The size of the error bars and the smoothness of the lines plotted in Fig. 8d for the hourly averaged mode 12 data indicate how accurately we can measure the Reynolds stresses in the present state of the art.

5. Discussion and conclusions

The two datasets analyzed herein indicate that mode 12 at a ping rate of 10 Hz, recording ensemble averages at 1 Hz, reduces the floor level for detection of Reynolds stress to about 0.02 Pa compared with about 0.06 Pa using mode 1 at 2-Hz recording ensemble averages at 0.5 Hz. At higher stresses, the uncertainty due to flow-related noise is reduced by using a higher number of ensembles for each estimate; for example, for a stress estimated at 1 Pa, the uncertainty in mode 12 is about 12% compared with 22% in mode 1. There is a similar effect on the detection of lower levels of TKE production: the detection level in mode 12 is approximately $5 \times 10^{-5}$ W m$^{-3}$, compared with $2 \times 10^{-4}$ W m$^{-3}$ in mode 1, an improvement by a factor of about 4. At higher production rates there is a reduction in the uncertainty from over 19% to less than 13%. In both the TKE production rate and the Reynolds stress, the reduction in the uncertainty is due to a combination of an increase in the number of pings per ensemble and an increase in the number of ensembles used to calculate the Reynolds stress.

a. Turbulent time scales

Consideration of the factor $\gamma$ [Eq. (8)] enables us to propose strategies to reduce the uncertainties in the estimates of Reynolds stress, shear, and TKE production rate. As the interval between measurements, $\Delta t$, is reduced to the autocovariance time scale of the turbulence, further increases in the frequency of velocity measurements will do little to decrease the uncertainty in the stress estimates due to the associated increase in $\gamma$ as adjacent measurements become increasingly covariant. The number of measurements $M$ that can be used to calculate the stresses is also constrained by the stationarity period of the flow. In tidal flows, the period $T$, over which the flow can be thought of as quasi-stationary, has been estimated at between 8 and 12 min (Soulsby 1980); analysis of the present datasets using a run test (Bendat and Piersol 1971) indicates that the datasets are quasi-stationary over the 10-min averaging period used over most of the tidal cycle, except for a short time of high acceleration at slack water. The stationarity period gives a maximum value of $M = T/\Delta t$, which for $T = 10$ min and $\Delta t = 1$ s sets a limit of $M \approx 600$. At present, the highest possible frequency of recording of velocity measurements using mode 12 is 1 Hz. We are therefore constrained at present by a maximum value for $M = T$ (s), even in regimes in which the turbulent time scales are shorter than 1 s. Using the maximum possible value of $M$ combined with the maximum number of pings in each ensemble will give the most accurate measurement of Reynolds stress for any given ADCP configuration.

If it were possible to record velocity measurements at a higher frequency, the best quality data would be obtained with an interval between measurements, $\Delta t$, which is equal to the autocovariance time scale of the turbulence. This would give the maximum possible value for $M$, considering the stationarity period, and would reduce the value of $\gamma$ to 1, with a consequent decrease in the standard deviation of the Reynolds stress estimates. By using an interval of less than $\Delta t$ between measurements we would expect an increase in $\gamma$ that would offset any improvement in the uncertainties due to the increase in $M$.

The correction factor $\gamma_b$ for estimates of $b_i^2$ is similar for mode 1 and mode 12, and somewhat smaller than those observed by Stacey et al. (1999), with a maximum value of around 2.5, and a median value of 1.6 for mode 1 and 1.7 for mode 12. The correction factor $\gamma_z$ for the shear $\bar{\partial u}/\partial z$ is approximately 1.5 for mode 1 and for the
higher-depth cells in mode 12. Closer to the bed in mode 12, the value of $\gamma_s$ increases to 2.9 on the flood and 2.3 on the ebb. This compares with a maximum value of $\gamma_s$ in mode 1 of 1.7. However, the largest values are only found around slack water, particularly on the transition from ebb to flood; at other times, the mode 12 values are comparable to the mode 1 values. This apparently longer persistence in the turbulent eddies may be due to the fact that the water is almost stationary, and hence the instrument is sampling the same eddy for a longer period, rather than different eddies as they are advected past the instrument.

If the time scale of the turbulence were greater than 1 s, we might expect that the correction factor $\gamma$ for the 1-s averages used in mode 12 would be higher than those for the 2-s averages used in mode 1, as in the case of the near-bed shear estimates. If the time scale were shorter than 1 s, there would be no significant correlation between one measurement and the next using either 1- or 0.5-Hz sampling frequencies. In the present datasets, we know there is a significant correlation, since both $\gamma_s$ and $\gamma_r$ are greater than 1. However, most values are relatively low, so the correlation between successive measurements can be assumed to be very weak, indicating that for the flows considered here we have reached a near-optimum sampling frequency.

**b. Strategies for improved sampling**

Further tests were carried out at the quarry using mode 12 with 20-ping ensembles. The along-beam velocity standard deviation for an ensemble mean at this ping rate was estimated as 0.0126 m s⁻¹. This gives a theoretical noise level in the TKE production rate of $\sim 1.5 \times 10^{-3}$ W m⁻² at zero flow, indicating that using a 20-Hz ping rate should give a reduction in the noise floor of the TKE production rate measurements of at least an order of magnitude over mode 1 data using a ping rate of 2 Hz and ensemble averages recorded every 2 s and a reduction by a factor of 3 compared to the 10-Hz measurements. The main disadvantage of using such a high ping rate is that it will reduce the maximum possible profiling range since each sub-ping must travel back and forth between the ADCP and back before the next one is sent. RDI (2002) also recommend a minimum time between sub-pings of 0.03 s for a 1.2-MHz ADCP. The range can be calculated from

$$t - 0.2 = \frac{2nr}{c} + an,$$

where $t$ is the ensemble time, $n$ is the number of sub-pings per ensemble, $a$ is the time between sub-pings, $r$ is the slant range, and $c$ is the speed of sound in water. The ADCP processing overhead is $\sim 0.2$ s per ensemble, which must be subtracted from the required ensemble time. A mode 12 ping rate of 20 Hz, with velocities recorded at 1 Hz, gives a maximum range of 7.5 m. When using a mode 1 or a mode 12 ping rate of 10 Hz, the range for a 1.2-MHz ADCP is not constrained by the ping rate, giving a range of around 15 m. Care should be taken when using very high ping rates to avoid ping-to-ping interference (RDI 2002).

If the autocovariance time scales were long enough, preaveraging the 10-Hz mode 12 data into 2-s ensembles would decrease the uncertainty by a factor of $\sqrt{2}$, with no significant loss of true variance due to turbulent fluctuations. However, for the York River dataset, where the autocovariance time scale is very short, this would also have the effect of underestimating the true variance. Rippeth et al. (2002) estimate that by recording at 0.5 Hz compared with 2 Hz, the variance is underestimated by $\sim 20\%$, indicating that this strategy is likely to be inadvisable.

**c. Effect of depth cell size**

The depth cell size chosen for the studies was 0.5 m in both cases. Near the boundary, as the length scale of the turbulent fluctuations decreases, some undersampling of the variance may result. Previous studies show this effect to be small: Lu et al. (2000) estimate that using a depth cell size of 1 m produces an underestimate of the Reynolds stress of 5% compared with a depth cell size of 0.1 m, and Rippeth et al. (2002) estimate a loss of less than 5% when comparing depth cells of 0.5 and 0.25 m. It should also be noted that an additional effect of decreasing the depth cell size is to increase the instrument noise, which may have serious implications in a low energy regime.

When configuring an ADCP for turbulence measurements, the choice of depth cell size, $\Delta z$, represents an irreversible decision in relation to stress and TKE uncertainty. Stress recorded with an initial choice of $\Delta z$ cannot be matched, in terms of statistical reliability, by averaging data from two adjacent depth cells of size $\Delta z/2$. Since the standard deviation of velocity varies approximately as $1/\Delta z$, we have

$$\sigma_{\gamma}(\Delta z/2) = 2\sigma_{\gamma}(\Delta z),$$

where $\sigma_{\gamma}(\Delta z)$ is the standard deviation of the along-beam velocities for a depth cell of size $\Delta z$. Averaging adjacent depth cells of size $\Delta z/2$ gives

$$\sigma_{\gamma}(2 \times \Delta z/2) = \sqrt{2}\sigma_{\gamma}(\Delta z),$$

indicating a higher standard deviation of the velocity measurements if the smaller depth cell size is selected, even after averaging over two depth cells. This gives the ratio of the uncertainty in the Reynolds stress estimates due to instrument noise using the two values of $\Delta z$:

$$\frac{\sigma_{\gamma}(2 \times \Delta z/2)}{\sigma_{\gamma}(\Delta z)} = \frac{\sigma_{\gamma}^2(2 \times \Delta z/2)}{\sigma_{\gamma}^2(\Delta z)} = 2,$$

since $\sigma_{\gamma} \approx \sigma_{\gamma}^* [\text{Eq. (14)}]$. If the stresses are calculated first, then averaged over two depth cells of size $\Delta z/2$, then the ratio is even less favorable ($2\sqrt{2}$).

In practice, in broadband ADCPs, $\sigma_{\gamma}^*$ does not vary
exactly as $1/\Delta z$, but exhibits a discontinuous variation with depth cell size, so that the optimal choice of $\Delta z$ may require reference to the manufacturers manual or configuration software.

d. Estimating uncertainties for future deployments

The estimate of the along-beam velocity standard deviation from measurements in a near-stationary water column or from RDI’s PlanADCP configuration software enables us to determine the floor level for detection of Reynolds stresses and TKE production rate. The floor level of detection of the parameters of interest can then be determined using Eqs. (14) and (19). It should be noted, however, that the RDI software tends to underestimate the standard deviation by a factor of about 1.3 for mode 1 and 1.7 for mode 12, compared to the stationary water column tests described here. The results presented here indicate that the fast-pinging mode 12 significantly improves estimates of turbulent parameters using the variance method.

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APPENDIX A

The Variance of the Reynolds Stress Estimates

Starting with Eq. (4) we have

$$\sigma_h^2 = \frac{1}{16 \sin^2 \theta \cos^2 \theta} \text{var} \left[ \frac{1}{M} \sum_{m=1}^{M} b_i^2(m) - \frac{1}{M} \sum_{m=1}^{M} b_i^2(m) \right] = \frac{1}{16 M^2 \sin^2 \theta \cos^2 \theta} \text{var} \left[ \sum_{m=1}^{M} b_i^2(m) - \sum_{m=1}^{M} b_i^2(m) \right].$$

Writing in a covariance form, we get

$$\sigma_h^2 = \frac{16 M^2 \sin^2 \theta \cos^2 \theta}{16 M^2 \sin^2 \theta \cos^2 \theta} = \frac{16 M^2 \sin^2 \theta \cos^2 \theta}{16 M^2 \sin^2 \theta \cos^2 \theta}.$$

Using the additive rule for covariances,

$$\sigma_h^2 = \frac{\sum_{i=1}^{M} \sum_{m=1}^{M} \text{cov}[b_i^2(m), b_i^2(n)] + \text{cov}[b_i^2(m), b_i^2(n)] - 2 \text{cov}[b_i^2(m), b_i^2(n)]}{16 M^2 \sin^2 \theta \cos^2 \theta}.$$

Rearranging into variances and covariances,

$$\sigma_h^2 = \frac{\sum_{i=1}^{2} \sum_{m=1}^{M} \text{var}[b_i^2(m)] + 2 \sum_{i=1}^{M} \sum_{m=1}^{M} \sum_{n=1}^{M} \text{cov}[b_i^2(m), b_i^2(n)] - 2 \sum_{m=1}^{M} \sum_{n=1}^{M} \text{cov}[b_i^2(m), b_i^2(n)]}{16 M^2 \sin^2 \theta \cos^2 \theta}.$$

APPENDIX B

The Variance of a Product

The variance of a product $xy$ is derived as follows (Mood et al. 1974):

$$\text{var}(xy) = E[(xy)^2] - [E(xy)]^2.$$

The mean value of the product, $E(xy)$ is given by

$$E(xy) = \bar{x} \bar{y} + \text{cov}(x, y).$$

Squaring this, we get

$$[E(xy)]^2 = \bar{x}^2 \bar{y}^2 + 2 \bar{x} \bar{y} \text{cov}(x, y) + [\text{cov}(x, y)]^2.$$

The mean value of the product squared is given by

$$E[(xy)^2] = E[(x' + \bar{x})(y' + \bar{y})^2].$$

Expanding,

$$E[(xy)^2] = E(x'^2 y'^2) + \bar{y}^2 \text{var}(x) + \bar{x}^2 \text{var}(y) + \bar{x}^2 \bar{y}^2 + 2 \bar{x} \bar{y} \text{cov}(x, y) + 2 \bar{x} \bar{y} \text{cov}(x, y) + 4 \bar{x} \bar{y} \text{cov}(x, y).$$

Hence,

$$\text{var}(xy) = \bar{y}^2 \text{var}(x) + \bar{x}^2 \text{var}(y) + E(x'^2 y'^2) - [\text{cov}(x, y)]^2 + 2 \bar{x} \bar{y} \text{cov}(x, y) + 2 \bar{x} \bar{y} \text{cov}(x, y) + 2 \bar{x} \bar{y} \text{cov}(x, y).$$

If $x$ and $y$ are independent, then only the first three terms are nonzero and

$$\text{var}(xy) = \bar{y}^2 \text{var}(x) + \bar{x}^2 \text{var}(y) + \bar{x} \bar{y} \text{var}(x) \text{var}(y).$$
For two variables that are not independent, but have zero mean, only the third and fourth terms remain:

$$\text{var}(xy) = E(x'^2y'^2) - [\text{cov}(x, y)]^2.$$ 

The two parameters of interest, $\overline{-u'w'}$ and $\overline{\partial u/\partial z}$, are calculated from the velocities in different depth cells of the ADCP, so in the case of the ADCP noise levels at zero flow, these two quantities are expected to be uncorrelated, so we can use the simplified equation using the variances to estimate the uncertainty due to instrument noise:

$$\text{var}(xy) = \text{var}(x) \text{var}(y).$$

Using the product of the variances as an estimator of $E(x'^2y'^2)$ will tend to underestimate the value of $E(x'^2y'^2)$ if the two quantities are not independent. However, for two strongly correlated quantities, the covariance term $[\text{cov}(xy)]^2$ is of the same order as the product of the variances, which will counteract this effect by reducing the estimate of $\text{var}(xy)$.

REFERENCES


