ELEMENTARY SOLUTIONS TO LAMB'S PROBLEM FOR A POINT SOURCE AND THEIR RELEVANCE TO THREE-DIMENSIONAL STUDIES OF SPONTANEOUS CRACK PROPAGATION

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ABSTRACT

Certain exact solutions to Lamb's problem (the transient response of an elastic half-space to a force applied at a point) involve the computation merely of three square roots, and about ten arithmetic operations (+, −, ×, ÷). They arise when both source and receiver lie on the free surface. It is just these solutions which are needed in a method due to Hamano for obtaining the slip function (displacement discontinuity), as a function of space and time, for planar tension cracks and shear cracks which grow spontaneously with arbitrary shape. The solutions are described here in detail, for an elastic medium with general Poisson's ratio. They include perhaps the simplest-possible example of the $P$-wave.

INTRODUCTION

A thorough understanding of motions in an elastic half-space, subjected to an applied force, is an essential part of wave-propagation theory needed to interpret seismic waves. For this reason, half-space problems have been the subject of an enormous literature, beginning with Lamb's (1904) classic study of displacements set up by forces applied at a point and along a line on the free surface. Here, I give some new solutions, these being the horizontal motions of the free surface for a horizontal force applied as a step in time at a point also in the free surface. (Throughout this paper, the half-space is oriented with a horizontal free surface. Taking Cartesian axes with $x_3$ as the depth coordinate, into the half-space, the free surface is $x_3 = 0$.)

Whatever the value of Poisson's ratio, the new solutions (which augment the work of Pekeris, 1955; Chao, 1960; Mooney, 1974) are extremely simple to compute. However, these formulas would be only a minor curiosity if it were not for one very important application, in which speed of computation is essential. This application is suggested by Hamano's (1974) method for studying spontaneous crack propagation. Since it is the larger problem of crack propagation which has motivated the present study, I shall, in what follows, give a brief review of Hamano's method, before giving the simple solutions to Lamb's problem.

MOTIVATION

Within an infinite homogeneous elastic medium, initially at rest, suppose that a crack nucleates at time $t = 0$ and subsequently grows within the plane $x_3 = 0$. Then a useful representation of displacement $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ throughout the medium can be
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given as

\[ u_n(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi_1 \, d\xi_2 \, G_{np}(x, t - \tau; \xi_1, \xi_2, 0, 0) T_p(\xi_1, \xi_2, \tau). \]  

(1)

Here, \( G_{np}(x, t; \xi, \tau) \) is the Green function for the medium, being the \( n \)-component of displacement at \( (x, t) \) due to a unit impulse applied in the \( p \)-direction at position \( \xi \) and time \( \tau \). For purposes of computing \( G_{np} \), it is required that the whole plane \( x_3 = 0 \) be a traction-free surface. \( T(\xi_1, \xi_2, \tau) \) is the actual traction occurring on the whole plane \( x_3 = 0 \) of which the crack is a part. Equation (1) is described further, and proved, by Das and Aki (1977) and Aki and Richards (1980, their equation 2.43).

Intuitively, the above representation can be understood as replacing the actual (crack) source of radiation by a whole plane, separating the medium into two half-spaces. Into each half-space, waves are radiated due to the same tractions as those set up by the crack, applied over the half-space surfaces. From the space-time element \( d\tau \, d\xi_1 \, d\xi_2 \) there is an applied impulse of strength \( d\tau \, d\xi_1 \, d\xi_2 \, T_p(\xi_1, \xi_2, \tau) \) in the \( p \)-direction. If the displacement contribution to \( u_n(x, t) \) from this element is to be considered in isolation from tractions acting elsewhere on \( x_3 = 0 \), then the appropriate Green function \( G_{np} \) must be constrained by having zero traction over the surface of the half-space. Hence, it must be a solution to Lamb's problem.

Hamano (1974) pointed out that for shear cracks and tension cracks, a soluble scheme for the displacement discontinuity \([u]\), say, across the crack can be set up from equation (1) by considering the \( x \) position itself in the crack plane, \( x = (x_1, x_2, 0) \), and using symmetry properties of \( u \) and \( T \) across \( x_3 = 0 \) to constrain the displacement and traction on different parts of the plane of the crack.

For a general tension crack \([u] = [0, 0, u_3]\) and \( T = (0, 0, T_3) \) so that the only Green function needed is \( G_{33} \).

For a general shear crack, \([u] = [u_1, u_2, 0]\) and \( T = (T, T_2, 0) \) so that the only Green functions needed are \( G_{11}, G_{12}, G_{21}, \) and \( G_{22} \) (see Das and Aki, 1977, for a related study of two-dimensional cracks). The jump in \( u_3 \) is zero, because opposite faces of the crack remain in contact. \( T_3 \) is zero, because planar shearing cannot change the normal stress on the crack plane. Together with \( G_{33} \) for tension cracks, these five different Lamb problems/Green functions need be studied only for the case that both source and receiver lie in the free surface of the half-space. Hamano's method is important in offering the chance to study spontaneous crack growth for completely general shapes of planar cracking. This paper contributes to that goal, by showing that just these Green functions are almost trivially simple to compute. For completeness, a single integral is also given below, in terms of which the remaining four components, \( G_{13}, G_{23}, G_{31}, G_{32} \), can efficiently be computed.

**FORMAL STATEMENT OF PROBLEM, AND ITS SOLUTION**

In this section, explicit formulas are derived for \( G_{np}^H(x_1, x_2, 0; t; \xi_1, \xi_2, 0, 0) \), this being the \( n \)-component of displacement at position \( (x_1, x_2, 0) \) and time \( t \), within the free surface \( (x_3 = 0) \) of a homogeneous, isotropic elastic half-space, due to a unit-step force in the \( p \)-direction applied also within the free surface, at position \( (\xi_1, \xi_2, 0) \), the step occurring at time \( 0 \). Once this solution has been found, \( G_{np} \) for an impulse (as required in the section above) is given by

\[ G_{np}(x, t - \tau; \xi, 0) = \frac{\partial}{\partial s} G_{np}^H(x - \xi, s; 0, 0) \big|_{s=t-\tau}. \]
Since related problems have had such a wide exposure, I shall abbreviate the description of how a solution is obtained. Thus, in general, the solution to Lamb’s problem for a point source can be obtained as an integral over just one variable. In our case,

\[
G^H(x_1, x_2, 0, t; 0, 0) = \frac{1}{\pi \mu r} \text{Imag} \left\{ \int_1^T \frac{H(T - 1)}{(T^2 - P^2)^{1/2}} \left[ \frac{LP dP}{(A - 2P^2)^{1/2} + 4YP^2} \right] \right\}
\]

where \( \mu = \text{rigidity}, \ r = (x_1^2 + x_2^2)^{1/2} \), \( H \) is the unit Heaviside step function, and capital letters in the integrand denote dimensionless quantities

\[
A = \alpha^2/\beta^2 \quad (\alpha = \text{P-wave speed}, \beta = \text{S-wave speed})
\]

\[
T = \alpha t/r \quad (T = 1 \text{ being the P-wave arrival time})
\]

\[
X = (1 - P^2)^{1/2} \quad \text{or} \quad -i (P^2 - 1)^{1/2}, \quad Y = (A - P^2)^{1/2} \quad \text{or} \quad -i (P^2 - A)^{1/2},
\]

\[
\text{Imag} \{ \} \quad \text{denotes the imaginary part of} \{ \}, \quad \text{and}
\]

\[
L_{11} = (T^2 - P^2)[2Y - 4X + (A - 2P^2)/Y] + AY \cos^2 \phi
\]

\[
- (T^2[2Y - 4X + (A - 2P^2)/Y] - AY) \sin^2 \phi
\]

\[
L_{12} = L_{21} = (T^2 - P^2)[2Y - 4X + (A - 2P^2)/Y] \cos \phi \sin \phi
\]

\[
L_{22} = (T^2 - P^2)[2Y - 4X + (A - 2P^2)/Y] + AY \sin^2 \phi
\]

\[
- (T^2[2Y - 4X + (A - 2P^2)/Y] - AY) \cos^2 \phi
\]

and

\[
L_{13}/\cos \phi = L_{23}/\sin \phi = -L_{31}/\cos \phi = -L_{32}/\sin \phi = T(A - 2P^2) - 2TXY
\]

where \( \phi \) is given by \( x_1 = r \cos \phi, x_2 = r \sin \phi \), so that \( \phi \) is the azimuth to \( x \).

Equation (1) can be written down from Johnson (1974, his equations 26 to 34, but using \( P^2 \) for his \( a^2t^2r^{-2} - a^2r^{-2} \)). Our variable \( P \) is a (dimensionless) horizontal slowness, and equation (1) is essentially a Cagniard solution in the form advocated by Helmberger (1968). Both source and receiver lie in the free surface, so the Cagniard path lies just above the real \( P \) axis in the complex \( P \) plane (see Figure 1, and legend).

In fact, the \( P \) integrals for \( G_{11}^H, G_{12}^H, G_{21}^H, G_{22}^H, \) and \( G_{33}^H \) can be given in closed form. This is not possible for the four remaining entries in \( G^H \), but these four are all proportional to just one integral which is still fairly simple to compute. We note first that

\[
G_{11}^H(x_1, x_2, 0, t; 0, 0) = \frac{[I_1(T)\cos^2 \phi - I_2(T)\sin^2 \phi]}{(\pi \mu r)}
\]

\[
G_{12}^H = G_{21}^H = \frac{[I_1 + I_3]\cos \phi \sin \phi}{(\pi \mu r)}
\]

\[
G_{22}^H = \frac{[I_3\sin^2 \phi - I_3\cos^2 \phi]}{(\pi \mu r)}
\]

\[
G_{33}^H = I_3(T)/(\pi \mu r)
\]

where arguments have been written out explicitly only for the first of equations (3).
FIG. 1. The Cagniard integration path for equations (2), (4), and (6) is shown as a solid heavy line. (a) Singularities of these integrands. They consist of branch cuts trending to the right along the real $P$ axis from points 1 and $\alpha/\gamma$, and a pole at $\alpha/\gamma$ ($\gamma$ being the Rayleigh wave speed). As well as the pole at $\alpha/\gamma = R_1^{1/2}$, we also show schematically the poles at $R_1^{1/2}$, $R_2^{1/2}$ in the right half-plane. $R_1$, $R_2$, and $R_3$ are roots of the Rayleigh cubic in $p^2$, $(A - 2p^2)^4 - 16X^2Y^2P^4 = 16(1 - A)(P^2 - R_1)(P^2 - R_2)(P^2 - R_3)$. For $A < 3.11$, $R_1$ and $R_2$ are real and lie between 0 and 1. But for $A > 3.11$, $R_1$ and $R_2$ are complex conjugates. Accurate locations for different $A$ are given in Figure 3b. (b) Path of integration in the vicinity of the Rayleigh pole. In the limit, as the semi-circle radius shrinks to zero, integration reduces to a principal value integration plus $-i\pi \times$ residue at the Rayleigh pole. The residue is imaginary from the integrands (4) of $I_1$, $I_2$, $I_3$, and hence gives zero net effect when the imaginary part is evaluated after multiplication by $-i\pi$. But the residue from integrand (6) for $I_4$ is real, leading to a non-zero contribution from the Rayleigh pole when $T > \alpha/\gamma$.

Just three functions of dimensionless time are needed to evaluate these five components of $G^H$, namely

$$I_1(T) = \frac{1}{\pi} \text{Imag} \left\{ \int_1^T \frac{H(T-1)}{(T^2 - P^2)^{1/2}} \left( \frac{(T^2 - P^2)[2Y - 4X + (A - 2P^2)/Y] + AY}{(A - 2P^2)^2 + 4XYP^2} \right) PdP \right\}$$

and

$$I_3(T) = \frac{1}{\pi} \text{Imag} \left\{ \int_1^T \frac{H(T-1)}{(T^2 - P^2)^{1/2}} \left( \frac{T^2[2Y - 4X + (A - 2P^2)/Y] - AY}{(A - 2P^2)^2 + 4XYP^2} \right) PdP \right\}$$

It follows from equation (3) that $I_1$ and $I_3$ are displacements within the vertical plane containing source and receiver, like the dominant motion in $P-SV$, whereas $I_2$ is a displacement transverse to this plane, like the dominant motion in $SH$. Analytic expressions are given below for each of these three integrals. A fourth dimensionless
solution to Lamb’s problem is introduced via

\[
G_{13}^H (x_1, x_2, 0; 0, 0) = I_4(T)\cos \phi / (\pi \mu r),
\]

\[
G_{23}^H = I_4 \sin \phi / (\pi \mu r),
\]

\[
G_{33}^H = -I_4 \cos \phi / (\pi \mu r),
\]

and

\[
G_{32}^H = -I_4 \sin \phi / (\pi \mu r).
\]

The integral for this solution is

\[
I_4(T) = \frac{1}{\pi} \text{Imag} \left\{ \int_1^T \frac{H(T - 1)}{(T^2 - P^2)^{1/2}} \left( \frac{T(A - 2P^2) - 2TXY}{(A - 2P^2)^2 + 4XYP^2} \right) PdP \right\}
\]

which cannot be given in closed form.

For a vertical point force, Pekeris (1955) obtained a closed-form solution for the vertical displacement \(I_3\) and a sum of elliptic integrals for the horizontal displacement \(I_4\). His solutions were restricted to the case \(\alpha^2/\beta^2 = 3\) (Poisson’s ratio = 0.25), and Mooney (1974) indicated how the evaluation of \(I_3\) might be carried out for any value of \(\alpha^2/\beta^2\) (though Mooney did not publish the solution formulas). For a horizontal point force, in a medium with \(\alpha^2/\beta^2 = 3\), Chao (1960) obtained closed-form solutions for the horizontal displacements \(I_1\) and \(I_2\) and a sum of elliptic integrals for the vertical displacement. Solutions themselves have not previously been given explicitly, for general \(\alpha^2/\beta^2\), for any one of the four basic (dimensionless) solutions \(I_i\).

In every case, the basic approach involves writing

\[
\frac{1}{(A - 2P^2)^2 + 4XYP^2} = \frac{(A - 2P^2)^2 - 4XYP^2}{(A - 2P^2)^4 - 16X^2Y^2P^4} = \frac{\text{Cubic}(P^2)}{\text{Cubic}(P^2)}
\]

so that the new denominator, of sixth order in \(P\), is real throughout the \(P\)-axis integration and has no branch cuts. The imaginary parts of the new integrals are easy to identify (together with a semi-residue contribution to \(I_4\) from indenting around the Rayleigh pole: see Figure 1 legend). If roots \(R_1, R_2, R_3\) are found for the cubic in \(P^2\), integration for \(I_1, I_2, I_3\) becomes possible using the partial-fraction decomposition

\[
\frac{1}{(A - 2P^2)^4 - 16X^2Y^2P^4} = \frac{a}{P^2 - R_1} + \frac{b}{P^2 - R_2} + \frac{c}{P^2 - R_3}.
\]

The Rayleigh pole lies at \(P = R_3^{1/2} = \alpha/\gamma\) (always on the real \(P\) axis, just to the right of \(P = \alpha/\beta\), since Rayleigh wave speed \(\gamma\) is a few per cent less than \(\beta\)). If Poisson’s ratio is less than a critical value, approximately 0.263, then \(R_1\) and \(R_2\) are real and lie between 0 and 1. But, for greater values of Poisson’s ratio, \(R_1\) and \(R_2\) are complex conjugates, as are \(a\) and \(b\) in equation (8), although \(c\) is always real. In this case there are poles in the complex \(P\) plane which can be associated with the so-called \(\bar{P}\) wave (Gilbert et al., 1962; Chapman, 1972; Aki and Richards, 1980), appearing between \(P\)- and \(S\)-arrivals. \(R_{1,2}^{1/2}\) and \(R_{3}^{1/2}\) lie on a Riemann sheet different from that which contains the Cagniard path.
Substitution of equation (8) into equations (7) and (4) leads to 24 integrable terms for each of $I_1$ and $I_2$, and six such terms for $I_3$. Extensive cancellation does eventually occur. The solutions, involving real positive constants $c_j (j = 1, \cdots, 7)$ and complex constants $C_k (k = 1, 2, 3)$, are as follows.

For times prior to and including the $P$ arrival, $T \leq 1$,

$$I_1 = I_2 = I_3 = 0.$$  \hfill (9)

For times between $P$ and $S$ arrivals, $1 < t < \alpha/\beta$, there are two different kinds of elastic media to consider. If Poisson’s ratio is less than 0.263 (corresponding to $A = a^2/\beta^2 < 3.11$),

$$I_1 = T^2 [c_1(T^2 - R_1)^{-1/2} - c_2(T^2 - R_2)^{-1/2} - c_3(R_3 - T^2)^{-1/2}]$$
$$I_2 = -c_4 + c_1(T^2 - R_1)^{1/2} - c_2(T^2 - R_2)^{1/2} + c_3(R_3 - T^2)^{1/2}$$
$$I_3 = c_4 - c_5(T^2 - R_1)^{-1/2} + c_6(T^2 - R_2)^{-1/2} - c_7(R_3 - T^2)^{-1/2}. \hfill (10)$$

If Poisson’s ratio is greater than 0.263, it is necessary first to define the complex square root (complex, because $R_2$ is then complex),

$$CROOT = \left[ (1 - R_2)(T^2 - R_2) \right]^{1/2} \hfill (11)$$

in which the choice of sign is made such that the complex number

$$1 + 2 \left( 1 - R_2 - CROOT \right)/(T^2 - 1)$$

has magnitude less than unity. Then,

$$I_1 = -T^2 \left[ \text{Real} \{C_1/CROOT\} + c_3(R_3 - T^2)^{-1/2} \right]$$
$$I_2 = -c_4 - \text{Real} \{C_2 \times CROOT\} + c_3(R_3 - T^2)^{1/2}$$
$$I_3 = c_4 + \text{Real} \{C_3/CROOT\} - c_7(R_3 - T^2)^{-1/2}. \hfill (12)$$

For times between the $S$ arrival and the Rayleigh-wave arrival, $\alpha/\beta < T < \alpha/\gamma$,

$$I_1 = 0.5 - 2c_3 T^2 (R_3 - T^2)^{-1/2}$$
$$I_2 = -2c_4 + 2c_3(R_3 - T^2)^{1/2}$$
$$I_3 = 2c_4 - 2c_7(R_3 - T^2)^{-1/2}. \hfill (13)$$

For times after the Rayleigh arrival, $\alpha/\gamma < T$,

$$I_1 = 0.5$$
$$I_2 = -2c_4$$
$$I_3 = 2c_4. \hfill (14)$$

Constants in the above solution need be evaluated just once for a given elastic medium, specified by the ratio $a^2/\beta^2$. An effective approach is first to find the largest root $R_3$ of the Rayleigh cubic; then to factorise $P^2 - R_3$ from the cubic and solve a quadratic for $R_1$ and $R_2$. Constants $a$, $b$, $c$ in equation (8) are given by

$$a^{-1} = 16(A - 1)(R_1 - R_2)(R_3 - R_1)$$
$$b^{-1} = 16(A - 1)(R_1 - R_2)(R_2 - R_3)$$
$$c^{-1} = 16(A - 1)(R_3 - R_1)(R_2 - R_3). \hfill (15)$$
Then
\[
\begin{align*}
    c_1 &= -2aA(A - R_1)(1 - R_1)^{1/2} \\
    c_2 &= 2bA(A - R_2)(1 - R_2)^{1/2} \\
    c_3 &= -2cA(R_3 - A)(R_3 - 1) \\
    c_4 &= A/(8A - 8) \\
    c_5 &= -2aAR_1(1 - R_1)(A - R_1)^{1/2} \\
    c_6 &= 2bAR_2(1 - R_2)(A - R_2)^{1/2} \\
    c_7 &= -2cAR_3(R_3 - 1)(R_3 - A)^{1/2} \\
    c_8 &= bA(2 - 2R_2)^2(1 - R_2).
\end{align*}
\]

Solutions given above for \(I_1, I_2, I_3\) require at most the evaluation either of three real square roots, or (depending on Poisson’s ratio) the evaluation of one complex square root and one real square root. These (worst) cases occur only between \(P\) and \(S\) arrivals. In terms of these closed-form solutions, all the five components of \(G^H\) relevant to Hamano’s method for studying spontaneous shear and tension cracks can be rapidly computed via equation (3).

Although \(I_4\) can be given in terms of elliptic integrals (with complex arguments when Poisson’s ratio is greater than 0.263), it is probably more efficient directly to integrate as follows

\[
I_4(T) = \begin{cases} 
0 & \text{for } T < 1; \\
\frac{2bA}{\pi} \int_0^{\pi/2} \frac{(P^2 - 1)(A - P^2)(A - 2P^2)}{(A - 2P^2)^4 - 16X^2Y^2P^4} \, d\chi \\
& \text{for } 1 < T < \alpha/\beta, \\
\frac{2TA}{\pi} \int_0^{\pi/2} \frac{(P^2 - 1)(A - P^2)(A - 2P^2)}{(T^2 - P^2)^{1/2}[A - 2P^2] - 16X^2Y^2P^4]} \, d\psi \\
& - \frac{H(T - \alpha/\gamma)c_8T}{(T^2 - R_3)^{1/2}} \quad \text{for } \alpha/\beta < T, \\
\end{cases}
\]

where \(P^2 = (T^2 - 1)\sin^2 \chi + 1\).

Integrals with respect to \(\chi\) and \(\psi\) here have well-behaved integrands. Note that, at time \(T = \alpha/\gamma = R_3^{1/2}\), a singular Rayleigh wave arrives (see Figure 1b legend) with strength proportional to the positive real constant

\[
c_8 = \frac{1}{2} cA(A - 2R_3)^3/R_3.
\]

Figure 2 shows the time-dependences of \(I_1, I_2, I_3\) for four different values of \(\alpha^2/\beta^2\). We note the following basic properties: (a) Displacements \(I_1\) and \(I_2\) are continuous across the \(P\)-arrival, as are \(I_2\) and the particle velocity \(dI_2/dT\). These results follow from equations (10) and (12), and relations between constants appearing in these formulas. (b) \(I_1\) and \(I_3\) are continuous across the \(S\) arrival, but have discontinuous slopes, whereas \(I_2\) itself is discontinuous. (c) \(I_2\) is continuous across the Rayleigh-wave arrival time, but \(I_1\) and \(I_3\) are singular. All three solutions are exactly constant after the Rayleigh singularity: these constants must then be the static solutions. (d) For the horizontal displacement due to a horizontal force, the step (in \(I_2\)) at the \(S\) arrival can be seen from equation (3) as having the orientation of an \(SH\) wave,
whereas the singularity (in $I_1$) at the Rayleigh arrival occurs as $P$-SV motion. However, because $P$-wave motion is not in general exactly longitudinal, the transverse motion (given by $I_2$) does begin at the $P$-wave arrival. (e) A $P$ wave is apparent in $I_3$ at times between and $P$ and $S$ arrivals, becoming more apparent with increasing values of $a^2/\beta^2$. Since it arises from a single algebraic expression, the term Real($C\sqrt{C}$) in equation (12), detailed properties of this wave are easy to investigate.

In Figure 3, the time-dependence of $I_4$ for four different values of $a^2/\beta^2$ is shown. Romberg integration was used, requiring occasionally up to 128 intervals for 1 per cent accuracy. There is a discontinuous slope at the $P$ and $S$ arrivals; a jump to a singularity at the Rayleigh arrival; and thereafter a gradual decay to the static limit.

CONCLUSIONS

Perhaps the main achievement of this paper is the exact form of constants $c_1, \ldots, c_6$ (positive real, if used) and complex constants $C_1, C_2, C_3$, in terms of which the complete solution to Lamb's problem can be given for any orientation of applied force, any displacement component, and any value of Poisson's ratio, provided both source and receiver lie in the free surface.

Four scalar solutions in $G$, involving the cross-terms (vertical or horizontal displacements due, respectively, to horizontal or vertical applied force), cannot be given in closed form, but a well-behaved integral solution is possible in general.
Fig. 3. (a) Values of the fundamental solution $I_4$ as a function of time. A closed-form solution is not possible in this case. Computation is for four different ratios of $a^2/\beta^2$. Dotted lines show values of $15 \times I_4$. (b) Since $T$ can be regarded as a value of $P$, the dimensionless horizontal slowness, we have here shown the complex $P$ plane with the same scale as the $T$ axis in (a), and Figure 2, a, b, and c. Values of $15 \times I_4$ are repeated from (a). Singularities $R_{1/2}, R_{2/2}, R_{3/2}$ here, for different values of $a^2/\beta^2$, occur then at times ($P$ values) which are indicative of what turn out to be properties of the $P$ and Rayleigh waves. Thus, $R_{1/2}$ for $a^2 = 2\beta^2$ is almost coincident with the ordinary $P$-wave arrival, making the latter highly impulsive for $I_1$ and $I_2$ because of the term in $(T^2 - R_2)^{-1/2}$. At larger $a^2/\beta^2$, the occurrence of complex $R_{2/2}$ with real values greater than 1 leads to the emergent broad pulse between $P$ and $S$ arrivals in $I_3$ and $I_4$. It is interesting that such a $P$ wave is not apparent for $I_1$ and $I_2$.

In the case of horizontal displacements due to a horizontally applied force, the solutions are relevant to a method for studying spontaneous shear cracks. For the case of vertical displacement due to a vertical force, the solution has relevance to tension cracks. In both these cases, solutions given by (3) and (9) to (14), for a step-applied force, are so simple that the following can readily be derived in closed form: (a) solutions for an impulsively applied force; (b) solutions averaged over $(r, r + \Delta r)$, $(\phi, \phi + \Delta\phi)$, and $(t, t + \Delta t)$; (c) 14 of the displacement fields $\partial G_{np}/\partial \xi_q = G_{np,q}$ due to a single-couple. Specifically, we can use reciprocity on $G_{np}$ so that the derivative is conducted with respect to receiver coordinates. From the five closed-form solutions in $G$, 10 single-couple displacement fields can be obtained by differentiating in the 1- and 2-directions, parallel to the free surface. The solutions for $(n, p, q) = (1, 3, 3), (2, 3, 3), (3, 1, 3)$, and $(3, 2, 3)$ can also be recovered in closed form, by using the linear strain constraints at a stress-free surface.

The simplicity of the five scalar solutions in $G$, which are associated with Hamano's method for studying cracks, is so remarkable that it gives high hopes of
successful development of a 3-dimensional study of spontaneous shear and tension fractures.

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