The Gill Model and the Weak Temperature Gradient Approximation

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(Manuscript received 15 March 2001, in final form 6 August 2002)

ABSTRACT

The authors investigate the accuracy of the weak temperature gradient (WTG) approximation, in which the divergent flow is computed by an assumed balance between adiabatic cooling and diabatic heating, for a prototype linear problem, the Gill model of a localized tropical heat source in a zonally periodic domain. As in earlier work by Neelin using a realistic, spatially distributed forcing, WTG is found to be a reasonable approximation to the full Gill model even when fairly large values of the thermal damping are used. Use of the local forcing and consideration of the different dispersion relations of free modes in the WTG and full Gill systems helps to clarify differences between the two solutions. WTG does not support an equatorial Kelvin wave. Instead, the Kelvin wave speed can be regarded as infinite in WTG. Hence, WTG becomes highly accurate when the thermal damping is sufficiently weak (0.1 day^{-1} or less) that an equatorial Kelvin wave can propagate around the globe without substantial loss of amplitude. Free Rossby waves are not equatorially trapped under WTG, and this increases the magnitude of the response far from the equator.

1. Introduction

Due to the smallness of the Coriolis parameter in the Tropics, stratified adjustment by internal gravity waves is able to rapidly eliminate horizontal temperature gradients. Consequently, the neglect of horizontal temperature gradients has been used as a simplifying assumption in many idealized model studies of the tropical atmosphere. Sobel et al. (2001) referred to the systematic use of this assumption as the weak temperature gradient (WTG) approximation. WTG can be used to derive a balanced tropical dynamics, including both rotational and divergent components (Held and Hoskins 1985; Browning et al. 2000; Sobel et al. 2001; Majda and Klein 2003). Balance models such as the quasigeostrophic equations are central to our theoretical understanding of extratropical dynamics. We are exploring the utility of WTG for providing a comparable clarification of large-scale tropical circulations.

Here, we consider the Gill model under WTG. Gill's (1980) seminal paper described the large-scale circulation that forms in response to prescribed localized steady heating associated with deep tropical convection. He used a linear damped shallow-water equation model on an equatorial β plane that provided elegant analytical solutions for some particular heating distributions. Mathematically identical models can be derived by a variety of different physical assumptions (e.g., Neelin 1989; Wang and Li 1993; Yu and Neelin 1997).

Neelin (1988, hereafter N88) used the Gill model under WTG (without using that nomenclature) to model the zonally asymmetric component of the surface wind, forced by either the vertical velocity at the top of a nominal planetary boundary layer (PBL) or by the precipitation field from a general circulation model (GCM). His solutions agreed reasonably well with the surface wind from the GCM.

Here, we consider the same problem as N88 but with a localized forcing similar to that used originally by Gill (1980). The use of this forcing, and consideration of the way WTG alters the free mode solutions of the underlying equations, sheds some light on how alterations to the dynamics under WTG cause differences in the solutions, whose magnitudes we quantify. We expect this understanding to carry over to more complex problems having some common features, such as nonlinear relatives of the Gill problem (e.g., Gill and Phlips 1986; Schneider 1987; Hsu and Plumb 2000) to which we might wish to apply WTG.

A strength of WTG is in simplifying the determina-

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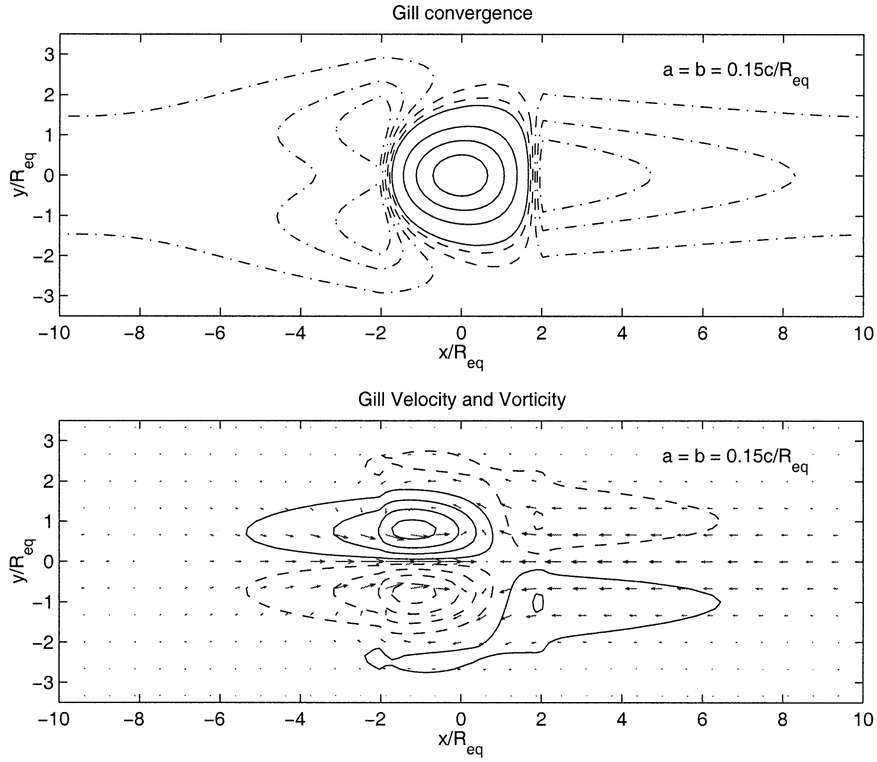

FIG. 1. Exact solution to the Gill model (no zonal compensation of mass sink) with zonally periodic boundary conditions. (top) Horizontal divergence (solid contours 0.1 to 0.9 by 0.2; dashed 0.02 and 0.06; and chain-dashed -0.1 to -0.02, by 0.04; all in units of D_0). (bottom) Velocity vectors and contours of vorticity (contour interval is $0.6D_0$, negative contours dashed). Full computational domain extends up to $|y/R_{eq}| = 10$.

tion of the spatial distribution of convective and radiative heating in a large-scale model of the Tropics. On the other hand, idealized models of tropical dynamics in which the location and/or magnitude of diabatic heating is specified are major components of our current understanding of the tropical atmosphere. The application of WTG to such models may not lead to major mathematical simplifications (e.g., Polvani and Sobel 2002). Nonetheless, we consider it useful to learn the strengths and limitations of WTG as applied to them. Doing so helps place into context other studies of aspects of the tropical circulation for which WTG is clearly advantageous, such as those involving single column modeling (e.g., Sobel and Bretherton 2000; Chiang and Sobel 2002) or the the idealized, nonrotating Walker circulation (Bretherton and Sobel 2002). In this manner, we hope to proceed in small steps toward a coherent balance theory for the entire tropical circulation.

2. Shallow water equation analysis

a. Equations

Gill's (1980) model consists of linear steady-state momentum equations on a β plane with a Rayleigh damping rate a:

$$-\beta y v = -\partial\phi/\partial x - au, \tag{1}$$

$$\beta y u = -\partial\phi/\partial y - av. \tag{2}$$

Here $\phi = g\eta$ is the geopotential perturbation associated with a downward gravitational acceleration g acting on a free-surface height perturbation η. The equation for the height perturbation with a specified "mass" (more precisely, volume) source $M(x, y)$ and a Newtonian cooling rate b is

$$b\phi + c^2(\partial_x u + \partial_y v) = gM(x, y) \tag{3}$$

Here, $c = (gH)^{1/2}$ is the shallow water wave speed computed from the equivalent mean layer depth H. For the first baroclinic mode of the unsaturated tropical troposphere $c \approx 50$ m s^{-1}. We also define the equatorial Rossby radius $R_{eq} = (c/\beta)^{1/2} \approx 1500$ km.

Gill took $b = a$ to permit analytical solution. Neelin (1988) Wang and Li (1993), and Wu et al. (2000), among others, have investigated the changes to the solution of the Gill model that can arise when these damping rates are not the same or the vertical structure of the heating and the atmosphere itself is more realistic.

To apply WTG, we approximate the free-surface height as horizontally uniform (i.e., $\phi = 0$) in the height equation (3) to derive a diagnostic equation for the horizontal divergence in terms of the heating:

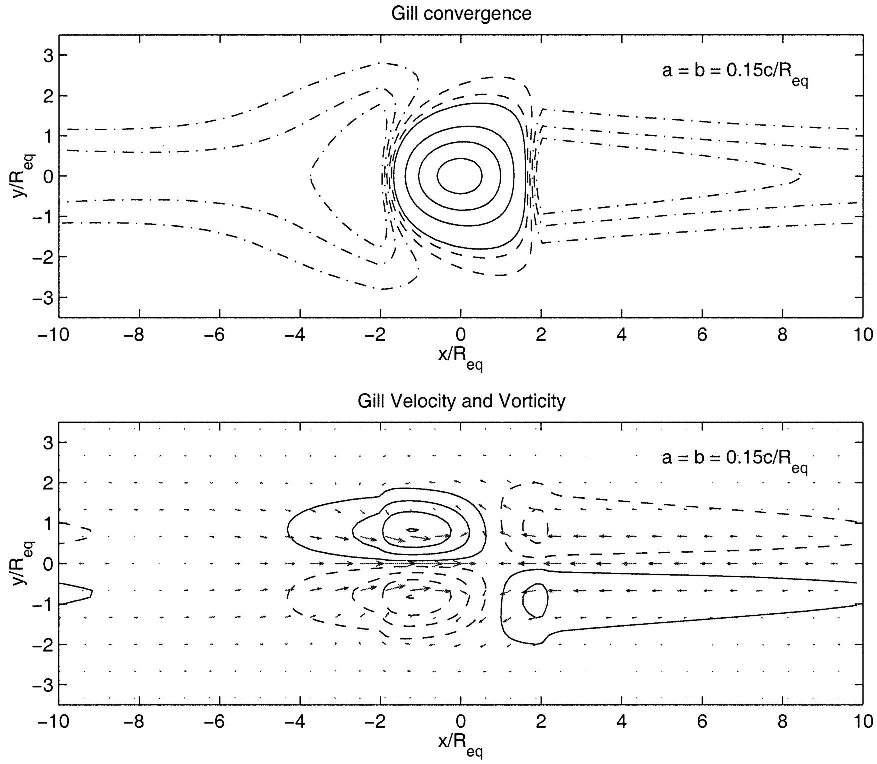

FIG. 2. As in Fig. 1 except with zonal compensation of the mass sink.

$$D = \partial_x u + \partial_y v \approx M(x, y)/H. \qquad (4)$$

WTG can be regarded as the limit $b = 0$ (no Newtonian cooling) of the Gill model. As discussed in the next two paragraphs, this limit is singular and has some complications. For this reason it has been avoided by past authors, with the exception of N88.

For application to the Tropics, we assume a zonally periodic domain of width $L = 40\,000$ km, and we look for a solution that decays to zero for large $|y|$. In Gill's original work, a localized mass sink (corresponding to convective heating lofting air out of the lower layer of the troposphere) was assumed. The solution to (1), (2), and (3) then implicitly determined a return flow driven by Newtonian cooling distributed around the heat source.

b. Compensation of the heating

The WTG divergence equation with only a localized mass sink is not consistent with the boundary conditions, as can be seen by integrating (4) over the entire domain in x and y. The integral of the left-hand side is zero, so a consistent WTG solution requires that we specify a compensating mass source to balance our localized mass sink. A subtle question arises regarding how precisely to distribute this mass source in space. WTG forbids us from using temperature (geopotential) perturbations to specify the compensating source, disallowing the Newtonian cooling parameterization used by Gill and subsequent studies. Since WTG in this context assumes that Kelvin waves act rapidly to homogenize ϕ and thus temperature gradients in longitude (see the discussion in section 3, and appendix), it is reasonable to assume that the compensating mass source is axisymmetric, but we must then still decide how to distribute it meridionally.

The meridional distribution of temperature, and thus radiative cooling in dry models that parameterize it as Newtonian, is determined by the Hadley circulation, which is fundamentally nonlinear for weak damping (Schneider and Lindzen 1977; Schneider 1977). An axisymmetric shallow-water nonlinear Hadley cell model under WTG was studied by Polvani and Sobel (2002), but we limit ourselves to linear dynamics here. (Study of WTG solutions in the nonlinear 2D case with both Hadley and Walker circulations is ongoing and will be reported in due course, but we expect insights gained from the linear case to be qualitatively relevant there.) As did N88, we therefore use a zonally compensated mass source, obtained by removing the zonal mean from the full mass source at each latitude and thereby removing any axisymmetric (Hadley) circulation. The zonally compensated forcing is the only one that has a well-behaved inviscid ($a = b = 0$) limit for the Gill problem (Schneider 1987), which is

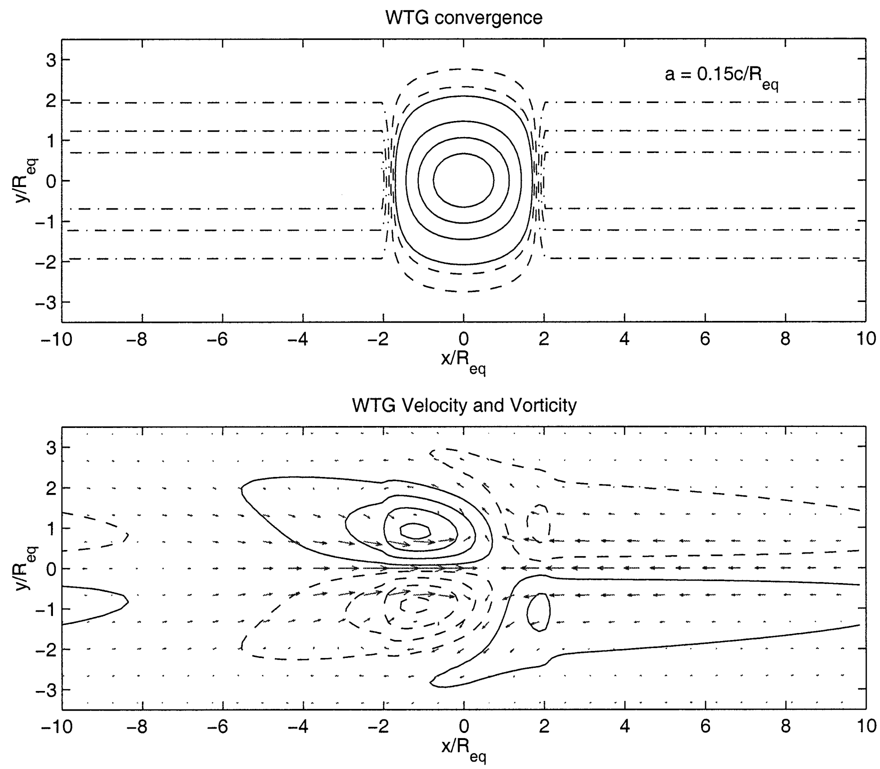

FIG. 3. As in Fig. 2 except with the WTG approximation.

$$v = -yM/H,$$
$$u = \int (2 + y\partial_y)M/H\,dx,$$
$$\phi = -\int \beta y^2 M/H\,dx, \tag{5}$$

expressing local Sverdrup balance (Gill 1980; N88).

For finite momentum damping rate a, alternate choices for the compensation are possible, and do not change the basic features of the zonally asymmetric part of the solution, as long as the total heating vanishes in an integral over the global Tropics, reasonably defined. One simple alternate choice, for example, is to compensate the localized mass sink with a mass source constant and nonzero over all longitudes in a specified latitudinal belt whose meridional scale is characteristic of the Hadley circulation, and zero outside that belt, as done in a somewhat different although related calculation by Schneider (1987). For this forcing, the solution is a superposition of the zonally compensated solution and a zonally symmetric overturning (''Hadley'') circulation with a strength inversely proportional to a. For the large value of a considered by Gill ($a \sim 0.5$ day^{-1}), the Hadley circulation is weaker than the zonally asymmetric circulation; for smaller a the assumption of linearity breaks down for the zonal angular momentum.

To apply WTG, we construct an equation for vertical vorticity $\zeta = \partial_x v - \partial_y u$ from (1) and (2), eliminating the divergence term using (4):

$$a\zeta = -\beta yD - \beta v. \tag{6}$$

The planetary vorticity advection term $-\beta v$ is still unknown, but v can be related to vorticity and divergence by noting that

$$\partial_x\zeta + \partial_y D = (\partial_x^2 + \partial_y^2)v. \tag{7}$$

Eliminating ζ in favor of v between (6) and (7), we obtain

$$[a(\partial_x^2 + \partial_y^2) + \beta\partial_x]v = (a\partial_y - \beta y\partial_x)D. \tag{8}$$

We numerically solve this elliptic boundary value problem for v using a fast Fourier transform in x and finite differencing in y, like Zebiak (1982) and N88. Knowing v, we can obtain ζ from (6) and u from the divergence. This only specifies u up to an y-dependent constant, which is determined by y integration of the zonally integrated vorticity equation. For zonally compensated heating, the zonal mean of u is identically zero.

In the WTG approximation, the perturbation geopotential field ϕ is then recovered from the divergence of the momentum equations:

$$aD - \beta y\zeta + \beta u = -(\partial_x^2 + \partial_y^2)\phi. \tag{9}$$

Boundary conditions on ϕ at the north and south boundaries of the numerical domain are found by setting $v = 0$ in the momentum equations:

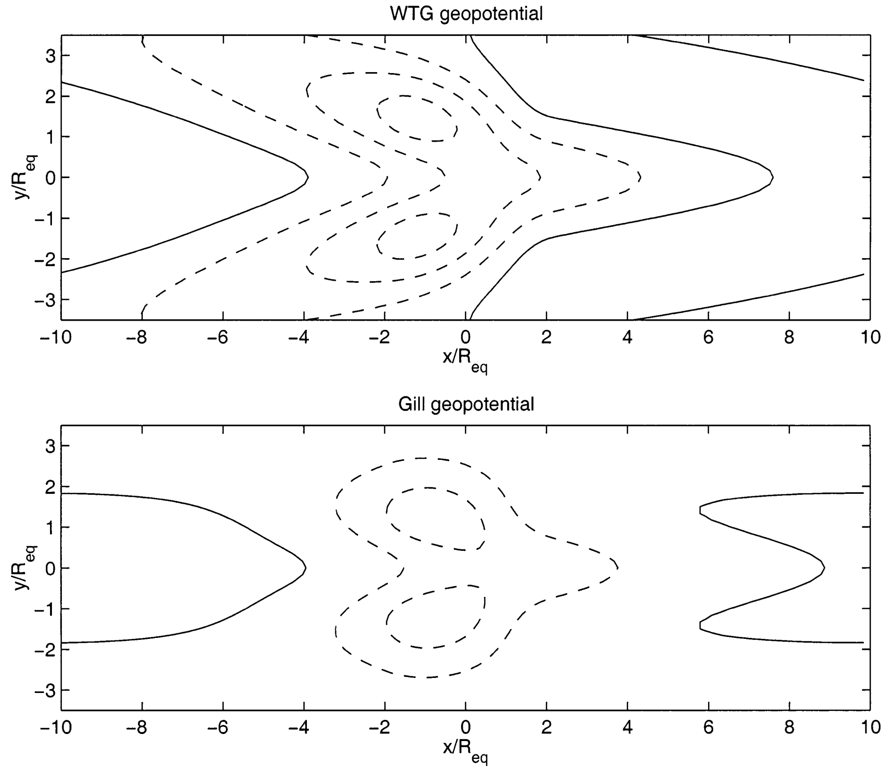

FIG. 4. Geopotential for the exact and WTG solutions (contour interval $0.4cR_{eq}D_0$, negative contours dashed).

$$(a\partial_y - \beta y\partial_x)\phi = 0 \quad \text{at} \quad y = \pm Y. \tag{10}$$

The full Gill model (1)–(3) can also be reduced to an equation for v:

$$[-b(a^2 + \beta^2 y^2)/c^2 + a(\partial_x^2 + \partial_y^2) + \beta\partial_x]v$$
$$= (a\partial_y - \beta y\partial_x)M/H. \tag{11}$$

Compared to (8), there is one additional term on the left-hand side that helps damp the dynamical response at large distances from the source. Gill (1980) found analytic solutions in an unbounded domain after making the longwave approximation (meridional geostrophic balance). For ease of comparison, we instead solve the full Gill model without approximation using the same domain and numerical technique as for WTG.

c. WTG dispersion relation

Gill (1980) interpreted his results in terms of damped equatorial Kelvin and Rossby modes spreading from the localized mass sink. It is illuminating to consider the corresponding WTG wave modes. To do this, we consider the unforced normal modes of the WTG model with time-dependent undamped momentum equations. The v equation of this model can be derived from (8) by replacing the damping a by the time derivative ∂_t and setting the divergence forcing to zero:

$$[\partial_t(\partial_x^2 + \partial_y^2) + \beta\partial_x]v = 0. \tag{12}$$

This equation admits normal modes of the form $v(x, y, t) = \hat{v}\exp[i(kx + ly - \omega t)]$ with a dispersion relation

$$\omega = -\beta k/(k^2 + l^2) \tag{13}$$

that is the same as for barotropic Rossby waves on a midlatitude β plane, although we emphasize that the flow due to these waves still has baroclinic vertical structure [see also Majda and Klein (2003)]. The deformation radius is effectively taken much larger than the wavelength, $(k^2, l^2) \gg R_{eq}^{-2}$, which causes the flow to be horizontally nondivergent in the absence of heating (Charney 1963). Equation (13) can be compared with the Rossby dispersion relation for the full time-dependent shallow-water equations, which is quantized into modes indexed by the index $n = 1, 2, 3, \cdots$:

$$\omega_n \approx -\beta k/[k^2 + (2n + 1)R_{eq}^{-2}], \tag{14}$$

where the approximation is fairly good for all wavenumbers and mode indices, and becomes exact in the limit $\omega_n \ll -\beta R_{eq}$ (Matsuno 1966). The WTG approximation does not trap the Rossby modes along the equator, so it allows a continuous spectrum of meridional wavenumbers. However, there is clearly a strong similarity between Rossby wave dispersion in the WTG and full Gill models. A particularly important similarity is that long zonal wavelengths in both have westward group velocity.

The full undamped time-dependent shallow-water

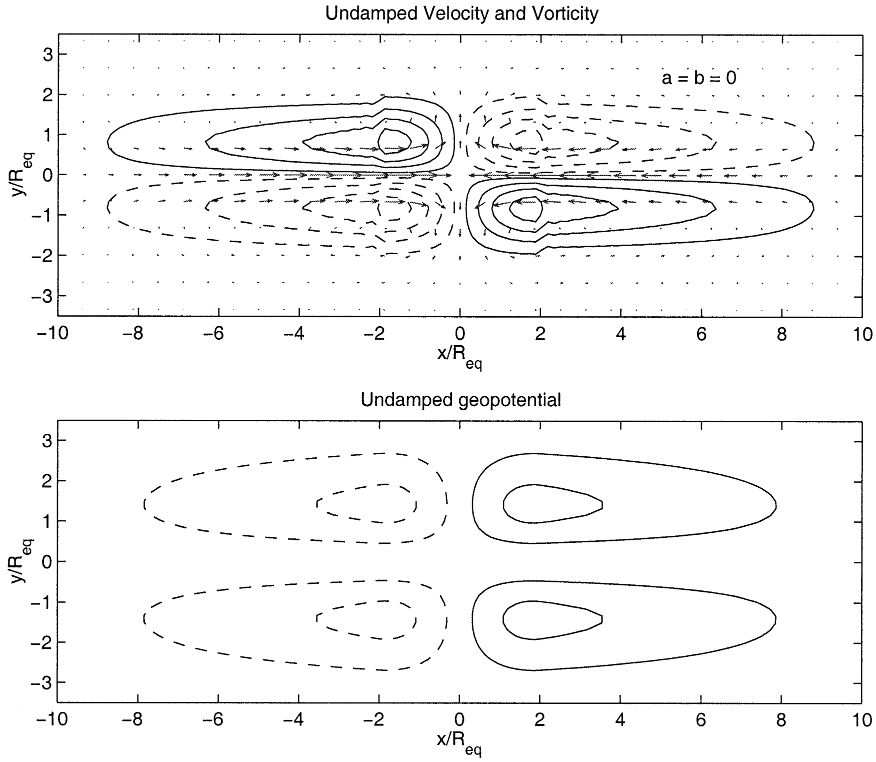

FIG. 5. Velocity/vorticity and geopotential for an undamped solution for the compensated mass sink of Fig. 2. In this case, the WTG approximation is an exact solution to the full Gill model, and the circulation is east–west symmetric about the mass sink (same contour intervals as in previous figures).

equations also support an equatorially trapped Kelvin wave with $v = 0$. This is not supported by the WTG approximation. Because the divergence is externally specified in WTG, the divergence of free wave modes must vanish, and the WTG normal modes all obey the polarization relation

$$k\hat{u} = -l\hat{v}. \tag{15}$$

A WTG normal mode with $v = 0$ and $u \neq 0$ must have $k = 0$, hence no zonal pressure gradient (again, justifying the axisymmetric compensation of the mass source if we imagine a thermal damping acting on an axisymmetric ϕ distribution). The zonal momentum equation then implies that it must also have zero frequency. This is just a special case of the Rossby dispersion relation. Thus the Rossby modes are the only free modes supported by WTG.

3. Results and discussion

Following Gill (1980), we consider a localized Gaussian mass sink

$$-\frac{M(x, y)}{H} = D_0 \exp\left(-\frac{(y - y_0)^2}{2R_{eq}^2}\right) \times \cos\left[\frac{\pi}{2}\min\left(\frac{|x|}{L}, 1\right)\right]. \tag{16}$$

The source is equatorially centered ($y_0 = 0$) with a width $L = 2R_{eq}$. We use a zonally periodic domain of width $20R_{eq}$ and north and south walls at $\pm 10R_{eq}$. The solutions presented here are insensitive to the meridional extent of the domain, as long as it includes the interval $|y| < 5R_{eq}$. The Rayleigh damping and Newtonian cooling rates are $a = b = 0.15c/R_{eq} \approx 0.5$ day^{-1}, essentially the same as shown in the plots of Gill (1980). A value of a this large is justified if the model is meant to simulate boundary layer flow as in N88. Both Rayleigh damping and Newtonian cooling rates are really much smaller in the free troposphere—where WTG is expected to be directly valid—on the order of 0.1 day^{-1} or less. While the right way to model the upper troposphere is to use small a and b in a nonlinear model, it is instructive, if not realistic, to consider the weakly damped linear solutions as well.

The divergence field and the vector velocity field (with vorticity contoured) for the full Gill model for this uncompensated mass sink are shown in Fig. 1. Because of the zonally periodic domain (and also because Gill made the long-wave approximation of meridional geostrophic balance, while we do not), the solution differs slightly from Gill's analytic solution but retains its essential features. The horizontal convergence closely follows the region of heating, while the divergence (highlighted using a smaller contour interval) has a Kelvin-

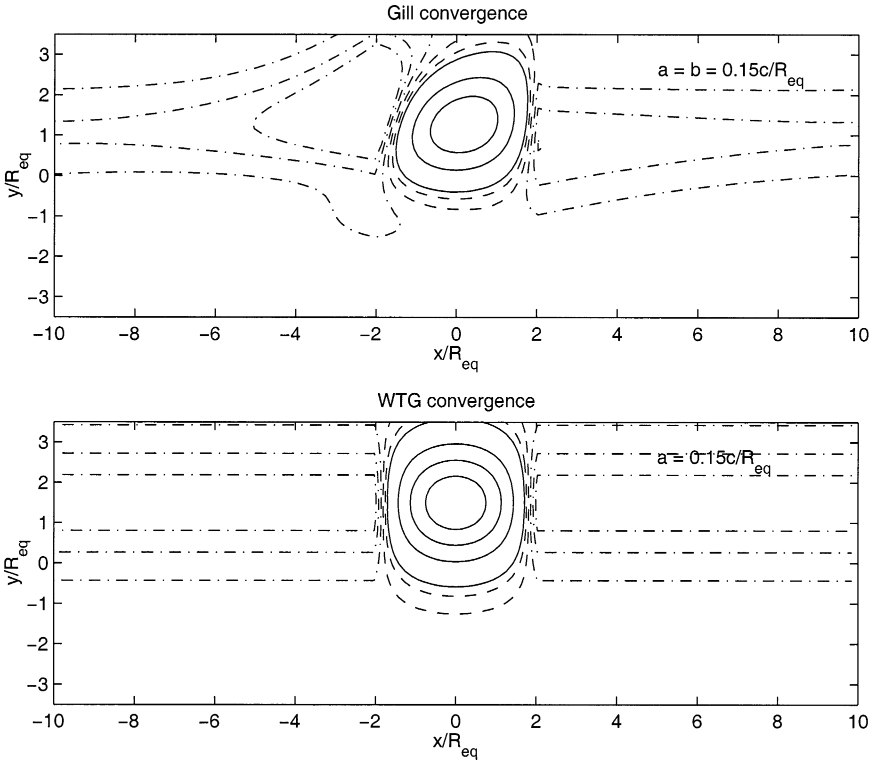

FIG. 6. Comparison of divergence for the standard Gill model and WTG given an off-equatorial zonally compensated Gaussian heat source centered at $y = 1.5R_{eq}$ (contours as in Fig. 2).

like structure to the east of the mass sink with a exponential decay length scale of c/a, wrapping around to merge with divergence in the west parts of the Rossby gyres. There is a weak Hadley circulation with zonally averaged convergence near the equator and zonally averaged divergence near the north and south edges of the heat source.

Figure 2 shows the corresponding fields for the full Gill model when the mass sink is zonally compensated:

$$M_c(x, y) = M(x, y) - \overline{M}(y), \tag{17}$$

where an overbar denotes a zonal average. The flow field is not greatly affected by the zonal compensation. The equatorial easterlies now extend further eastward from the mass sink and the zonally averaged Hadley circulation has been removed.

The same fields for the WTG solution with a zonally compensated mass sink are shown in Fig. 3. There is again no zonally symmetric circulation. The Gill and WTG models differ only in how divergence is obtained. The upper panels of Figs. 2 and 3 show that the divergence field is fairly similar in the two models, with slight reduction of the divergence in the Gill model well away from the mass sink. This provides a posteriori reassurance that WTG is a useful approximation for this mass sink.

The difference in any field $\psi(x, y)$ between the WTG approximation and the full zonally compensated Gill model can be quantified using a relative squared error:

$$\mathrm{RSE}(\psi) = \frac{\displaystyle\int_{-L/2}^{L/2} dx \int_{-Y}^{Y} dy[\psi_{\mathrm{WTG}}(x, y) - \psi_{\mathrm{gill}}(x, y)]^2}{\displaystyle\int_{-L/2}^{L/2} dx \int_{-Y}^{Y} dy \psi_{\mathrm{gill}}^2(x, y)}. \tag{18}$$

The divergence error due to making the WTG approximation has $\mathrm{RSE}(D) = 0.08$. This modest error can be regarded as the source of the errors in all other fields.

The velocity and vorticity fields, including the Rossby gyres and even the Kelvin-like response to the east are quite well reproduced by WTG [$\mathrm{RSE}(u) = 0.05$, $\mathrm{RSE}(v) = 0.25$, $\mathrm{RSE}(\zeta) = 0.16$], though they are slightly less equatorially trapped than in the full Gill model.

The Kelvin-like zonal wind pattern in the WTG solution is at first sight rather surprising since the WTG system only permits Rossby waves. The WTG easterly equatorial jet is not a manifestation of a damped wave, but is a response to the compensating zonally symmetric mass source. This induces an equatorially centered, zonally divergent flow with relatively weak meridional winds to the east of the mass sink. Because of this constant zonal divergence, the easterly equatorial jet decreases linearly, rather than exponentially, to the east

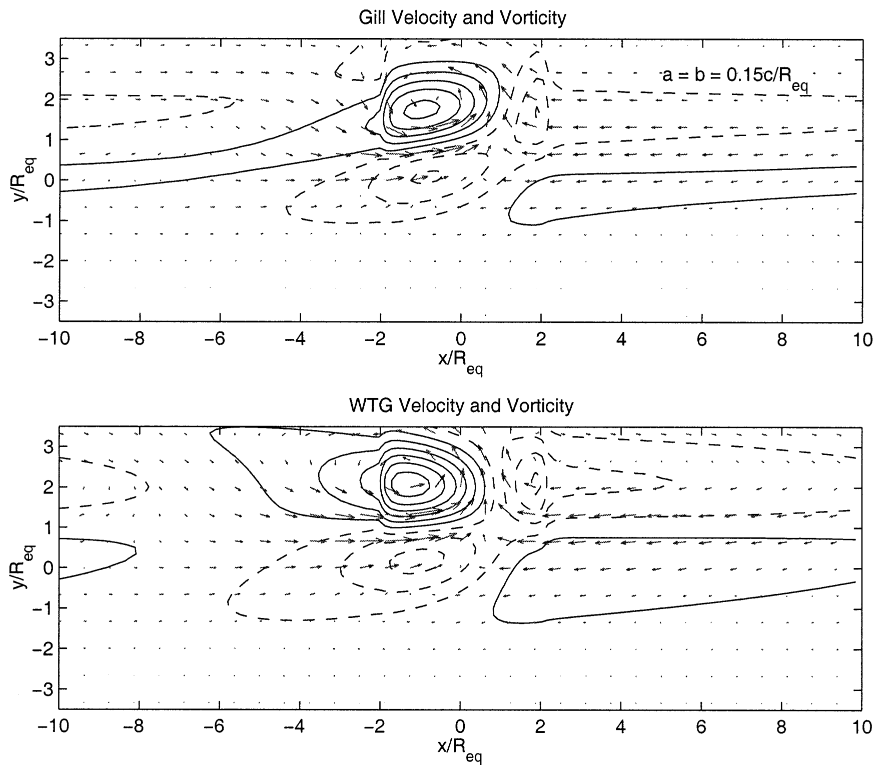

FIG. 7. Comparison of velocity (vectors) and vorticity (contours) for the standard Gill model and WTG given an off-equatorial zonally compensated Gaussian heat source centered at $y = 1.5R_{eq}$ (contours as in Fig. 2).

(this is also seen in the zonally compensated Gill solution). In contrast, the equatorial zonal velocity and divergence in a damped Kelvin wave would decrease exponentially with distance eastward of the mass sink.

However, consider a case in which the frictional damping a is small, as appropriate for the free troposphere, but the thermal damping b is initially large. As b is reduced within a periodic domain, the Kelvin wave can more nearly circumnavigate the entire tropical belt before being damped and produces a zonally uniform divergence along the equator. In this respect, the east–west symmetry in the WTG divergent flow is an asymptotic limit of the Gill solution. This qualifies the statement by N88 that this part of the response has ''nothing to do with a Kelvin wave since there is no $v = 0$ solution in the model.'' The model does not contain Kelvin waves, but the response has something to do with them; the WTG model is a ''fast Kelvin wave'' limit.

This discussion motivates the scaling argument in the appendix, which shows that for a zonally compensated mass source, the WTG solution is an accurate approximation of the full Gill model if $bL/c \ll 1$, that is, if an equatorial Kelvin wave can circumnavigate the globe without substantial damping of its amplitude. This dependence on c also implies that WTG is most accurate for the first baroclinic mode, and progressively less accurate for higher-order vertical modes, since they have smaller c.

A comparison of the Gill and WTG geopotential fields for the zonally compensated mass sink is shown in Fig. 4. For the WTG solution, ϕ is diagnosed from (9). For the full Gill model, ϕ is diagnosed by eliminating u between the x momentum and thermodynamic equations. The WTG ϕ qualitatively reproduces the full Gill solution, but is not nearly as equatorially trapped because of the modification to the Rossby wave dispersion relation by WTG discussed in the previous section. The amplitude of the WTG response is also larger overall due to the lack of any direct damping of ϕ itself in that solution. These differences result in a sizeable relative squared error RSE(ϕ) = 1.07. This error would be much reduced if a smaller, more realistic thermal damping b were adopted in the Gill model.

In particular, it is interesting to look at the limit of no damping ($a = b = 0$), when the zonally compensated Gill solution (5) is the same as the WTG solution, and the rotational circulation is in exact Sverdrup balance. Now the circulation, shown in Fig. 5, is east–west symmetric in v about the mass sink. Rossby waves can propagate around the globe undamped and no longer break this symmetry. With realistic upper-tropospheric thermal and Rayleigh damping timescales of 10 days, the circulation in the Gill model (not shown) approaches this regime, and is well approximated by WTG. However, this solution is unrealistic as a model of the upper troposphere due to the neglect of nonlinearity.

Lastly, we compare the WTG and Gill solutions for an off-equatorial zonally compensated Gaussian heat source of the form (16) centered at $y_0 = 1.5R_{\mathrm{eq}}$, with the same damping rates as in Fig. 1. Figure 6 shows that the WTG divergence field qualitatively matches the Gill solution. However, the Gill solution has considerably stronger equatorial trapping of the divergence away from the heat source. As in the equatorially centered case, the Gill solution has considerable asymmetry in the divergence between the two sides of the heat source driven by the Rossby response to the west and the Kelvin response to the east. By construction, this asymmetry is not present in the specified WTG divergence. The relative squared error of the WTG divergence, $\mathrm{RSE}(M) = 0.42$, is five times as large as for the equatorially centered heat source. This large error is to be expected since there is substantial heating, and response, in the extratropics ($|y| > 2R_{\mathrm{eq}}$) where WTG is no longer appropriate on planetary scales.

Figure 7 shows that the velocity and vorticity fields for the two solutions are also in qualitative agreement, with accentuation of the Northern Hemisphere vortex in both. As in the equatorially centered case, the WTG solution exhibits less equatorial trapping of the velocity field. The relative error of the WTG vorticity is 4.5 times larger than in the equatorially centered case [$\mathrm{RSE}(\zeta) = 0.73$], owing to the sizeable divergence errors.

4. Conclusions

The weak temperature gradient approximation was examined in the context of the Gill model of the equatorial steady-state damped shallow-water equations forced by a specified mass source. In this context, WTG amounts to ignoring thermal damping, so the divergence is strictly equal to the mass source. WTG preserves most features of the circulation driven by localized heating reasonably well. Both the rotational and divergent circulation near the heat source are accurately captured. Although WTG does not support Kelvin waves, the zonal compensation of the heat source does induce a Kelvin-like response in the WTG solution to the east of the source qualitatively similar to that seen in the Gill model. Baroclinic Rossby waves are the one free wave type supported by WTG but, unlike in the Gill model, they are not equatorially trapped. This results in a somewhat less equatorially trapped far-field response to a localized tropical heat source than in the Gill model. For both the exact and WTG solutions, the steady state will equilibrate to changes in the heating on a timescale no longer than the travel time of a Kelvin wave around the equator, which is less than 10 days. Hence, WTG should apply to specified heat sources varying on timescales exceeding this as well as to steady heat sources.

For the Gill problem, WTG does not provide dramatic mathematical or conceptual simplification. However, for the actual moist Hadley–Walker circulation, WTG dramatically simplifies calculation of horizontal divergence by permitting its determination on a column-by-column basis. We are continuing to work toward testing the strengths and limitations of WTG for modeling the entire moist quasi-steady tropical circulation, and possibly even its intraseasonal variability.

Acknowledgments. We acknowledge helpful discussions with Mark Cane, David Neelin, and the reviewers, NSF Grant ATM-0096195, NASA Grant NAGS5-10624, and a Packard Foundation Fellowship for Science and Engineering granted to AHS.

APPENDIX

Scaling Analysis of WTG for the Gill Model

Consider a localized mass sink with characteristic zonal and meridional scales L_x and L_y, with significant amplitude out to a distance Y from the equator. If the sink is equatorically centered, $Y = L_y$; otherwise $Y > L_y$. Suppose the characteristic divergence in the center of the sink is D_0, and we assume that the sink is zonally compensated. We use the notation $\{\cdot\}$ to denote ‘‘characteristic scale of.’’

From (3), the WTG divergence field will accurately represent the Gill divergence field near the mass sink if $b\{\phi\}/c^2 \ll D_0$. However, for WTG to get the far-field divergent circulation right, it is necessary that the Newtonian cooling term produce a divergence much less than the compensating mass source, which is a factor of L_x/L smaller than the localized mass sink. Thus, for WTG to give an accurate global divergence field we must have

$$b\{\phi\}/c^2 \ll D_0 L_x/L, \tag{A1}$$

since then the WTG divergence field will accurately represent the overall divergence field. Since dissipation is weak, we can scale ϕ from the no-dissipation limit (5). This implies that

$$\{\phi\} = \beta Y^2 D_0 L_x. \tag{A2}$$

Hence, WTG will be accurate if

$$b\beta Y^2 L/c^2 \ll 1. \tag{A3}$$

For an equatorially centered source of width $O(R_{\mathrm{eq}})$, we take $Y = R_{\mathrm{eq}}$. Then, regardless of the zonal dimension of the localized mass sink, we see that WTG will be accurate if

$$b\beta R_{\mathrm{eq}}^2 L/c^2 = bL/c \ll 1. \tag{A4}$$

Now, L/c is the time for an equatorial Kelvin wave to propagate around the globe. Hence, WTG will be asymptotically accurate in the limit that this Kelvin wave experiences negligible damping of its amplitude in a traversal of the globe. The timescale L/c will be larger for higher baroclinic modes with smaller c, so WTG will become less accurate for motions of smaller vertical scale.

Sobel et al. (2001) did a scaling analysis of WTG on an f plane. They found that, for linear dynamics, WTG is asymptotically accurate in the limit that the horizontal size of the mass sink is much smaller than a Rossby radius R and that $aR/c \ll 1$. However, this analysis is not easily transferable to the Gill problem since R varies dramatically with latitude on an equatorial β plane and because an f plane does not support the basic Sverdrup balance that we assumed to get ϕ.

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