Water vapor as an active scalar in tropical atmospheric dynamics

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Water vapor is a constituent of the tropical atmosphere which, though to a significant extent locally controlled by vertical advection, precipitation, and surface evaporation, is also affected by horizontal advection. Water vapor affects the flow in turn, because a humid atmosphere supports deep, precipitating convection more readily than a dry atmosphere. Precipitation heats the atmosphere, and this heating drives the flow. Water vapor is thus a dynamically active constituent. Simplifications to the primitive equations of dynamical meteorology, based on the so-called weak temperature gradient approximation, are presented which highlight this behavior. The weak temperature gradient approximation is valid on large scales near the equator. It eliminates gravity waves, leaving only balanced dynamics, though the fundamental balance occurs in the temperature rather than the momentum equation (as is customary in most balance models of geophysical fluid dynamics). The dynamical role of water vapor is examined in a couple of idealized contexts, where either the vertical or horizontal structure of the flow is severely simplified. [DOI: 10.1063/1.1480795]

The study of atmospheric and oceanic dynamics has benefited from the development of "balance models" in which fast inertia-gravity waves are eliminated. Such models, of which the prototype is the quasigeostrophic model, are economically expressed in terms of an active scalar, known as the potential vorticity. Models of this type have been useful primarily for describing the extratropical flow, as they assume that the divergent flow is negligible compared to the rotational flow and (relatedly) that diabatic heating is unimportant, both very questionable assumptions in the tropics. Here I describe a balance model for the tropical atmospheric flow based on more empirically appropriate assumptions. In this model the balance is between heating (variations of which are dominated by variations in precipitation) and vertical advection of potential temperature. Thus the precipitation largely controls the divergent flow, which in turn controls the rotational flow. Atmospheric water vapor is advected by both components of the flow, and in turn its distribution influences the distribution of precipitation. Water vapor is thus an active scalar in the tropical atmosphere, and its role as such is highlighted in the balance model presented here.

I. INTRODUCTION

Current theoretical understanding of the large-scale behavior of the *extratropical* atmosphere and oceans rests to a large extent on the concepts of balance and potential vorticity.[1] Reduced equation systems such as the quasigeostrophic model,[2] which express these concepts, can be derived from systematic approximations based on scaling arguments in the limit of strong rotation and stratification (small Rossby and Froude numbers) and weak viscosity and diabatic heating. These approximations eliminate fast inertia-gravity waves, leaving only slow rotational motions. The only remaining linear wave mode is the Rossby wave, whose restoring mechanism depends on the rotation and sphericity of the earth. The model equations can be phrased in terms of a scalar called the potential vorticity (PV), a Lagrangian invariant for adiabatic and inviscid flow. Through the assumption of a diagnostic, instantaneous relationship between the mass and momentum fields ("balance"), and given appropriate boundary conditions, the entire flow—both mass and momentum fields—can be obtained from the PV distribution by solving an elliptic equation. Thus the entire flow dynamics is reduced to the dynamics of a single scalar which controls its own evolution.

Here I discuss the *tropical* atmosphere, to which the notion of balance has historically been applied with rather less success than to the extratropical atmosphere. I will outline some recent theory which provides a somewhat new picture (though having many partial precedents in the literature) of balanced tropical dynamics. In this picture, PV is demoted in importance, but is partly replaced at the core of the dynamics by another advected scalar which plays an active role in controlling the flow: atmospheric water vapor. Primarily, the activeness of water vapor comes here through its effect on deep, precipitating convection, which heats the atmosphere. This heating drives the flow, which in turn controls the water vapor distribution.

Length constraints preclude a review of the basic physics of water vapor in the atmosphere, though this would be helpful to the nonspecialist reader. Most of the relevant information has been summarized in a recent review article.[3]

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II. BASIC EQUATIONS

We start by considering the "primitive" (though hydrostatically balanced) equations of dynamical meteorology.[4] The equations for potential temperature, specific humidity, horizontal momentum, and mass conservation are, in that order:

$$\frac{D\theta}{Dt}=Q_T, \tag{1}$$

$$\frac{dq}{dt}=Q_q, \tag{2}$$

$$\frac{d\mathbf{u}}{dt}+f\mathbf{k}\times\mathbf{u}=-\nabla\Phi+\mathbf{F}, \tag{3}$$

$$\nabla_H\cdot\mathbf{u}+\rho_0^{-1}\frac{\partial(\rho_0 w)}{\partial z}=0. \tag{4}$$

θ is the potential temperature, $\theta=T(p_s/p)^{R/c_p}$, with R the gas constant for dry air and c_p the heat capacity of dry air at constant pressure, p pressure and p_s a reference value of the surface pressure, q specific humidity (mass of water vapor per unit mass of air), $\mathbf{u}$ horizontal velocity, ∇ horizontal gradient on pressure surfaces, $\Phi=\int_0^{z^*} g\, dz^*$ the geopotential on a pressure surface (with z^* geometric height of the surface and g gravitational acceleration), $\mathbf{F}$ frictional forces, $\mathbf{k}$ vertical unit vector. The Coriolis parameter, $f=2\Omega\sin\phi$, with Ω the rotation rate of the earth and ϕ the latitude. The total derivative is defined

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$$

with $\mathbf{u}=(u,v,w)$ the velocity components. The set (1)–(3) is stated in a log-pressure vertical coordinate, which in practice is close to geometric height. The lack of a time derivative in the mass conservation equation (4) is a consequence of the coordinate surfaces' having constant pressure and the assumption of hydrostatic balance. ρ_0 is a basic state density, assumed to be a function of z only. To a reasonable approximation, by hydrostatic balance $\rho_0\sim\exp(-z/H)$ where the scale height $H=R\langle T\rangle/g$ with $\langle T\rangle$ a layer mean temperature over the troposphere and R the gas constant for air. In everything that follows we will make the β-plane approximation, in which the horizontal coordinate is taken Cartesian and $f=f_0+\beta y$, with β, f_0 constants. At times we will consider either the "f plane," $\beta=0$, or the "equatorial β plane," $f_0=0$.

The right-hand sides of Eqs. (1) and (2) represent sources and sinks of heat and moisture due to unresolved processes, such as radiative transfer of electromagnetic energy, or to processes that are ultimately fluid-dynamical but are "sub-gridscale" with respect to some smoothing filter that is implicitly understood to have been applied to the equations, leaving second-order correlation terms in the filtered scales.[5] We can write

$$Q_T=Q_c+Q_R+Q_{\mathrm{diff}},$$

where Q_R represents radiative heating or cooling, and Q_c is the apparent source of heat associated with buoyant moist convection. Q_c thus represents release of latent heat by condensation of water vapor or freezing of liquid water as well as subgridscale transports (generally considered to occur primarily in the vertical direction) of heat. Q_q represents subgridscale transports of water vapor as well as loss by condensation.[5] Q_{diff} represents diffusive or turbulent transport by motions not directly associated with deep convection. We will assume that this term is only important in a shallow planetary boundary layer (PBL) near the surface.

In addition to the interior source and sink terms which appear in Eqs. (1) and (2), there are boundary fluxes of heat and moisture, referred to as the surface sensible heat flux and the surface latent heat flux or (equivalently) surface evaporation. Over the tropical oceans, the surface sensible heat flux is relatively small and can be neglected compared to the surface evaporation.

To obtain a closed dynamical system, we need to *parameterize* the source terms Q_c, Q_R, Q_q, etc., as functions (or functionals) of the large-scale state variables T, q, $\mathbf{u}$. Q_c and Q_q in particular are determined by a *convective parametrization*,[6] often the source of some controversy. Space precludes a detailed discussion of the parametrization problem here, but a few useful textbooks and review volumes exist.[7–9] For our purpose the key property of most modern convective parametrizations is that they depend to a large degree (sometimes exclusively) on the local vertical profiles of T and q at a given horizontal location, as these variables determine the stability of the atmosphere to moist convection.

III. TROPICAL DYNAMICS

In the extratropics, f is large enough and the flow weak enough that the Rossby number for large-scale flow is often small,

$$\mathrm{Ro}\equiv\frac{U}{fL}\ll 1,$$

with U, L the velocity and length scales. The dominant balance in the momentum equation is then geostrophic:

$$f\mathbf{k}\times\mathbf{u}=-\nabla\Phi. \tag{5}$$

Using this and a couple of other assumptions as the basis of a consistent scale analysis, the quasigeostrophic equations can be derived.[2,4] These equations have only rotational solutions, excluding inertia-gravity waves. They allow the entire flow to be specified by the self-determined evolution of a single scalar, the potential vorticity, given appropriate boundary conditions.[1] In addition to small Ro, the other key assumptions are that the Froude number (ratio of inertia to stratification effects) be small, that the meridional length scale of the motions not be too large ($\beta L\ll f_0$), and that the right-hand side terms Q_T and $\mathbf{F}$ be small. If Q_T can be entirely neglected, the entire moisture equation (2) can be neglected also, because it is only through Q_T that q can influence the flow.

Close to the equator, quasigeostrophic theory fails. Geostrophic balance expressed by Eq. (5) still holds to some degree, but as $f\rightarrow 0$ as one approaches the equator, $\nabla\Phi\rightarrow 0$ also, because no other term in the momentum equation

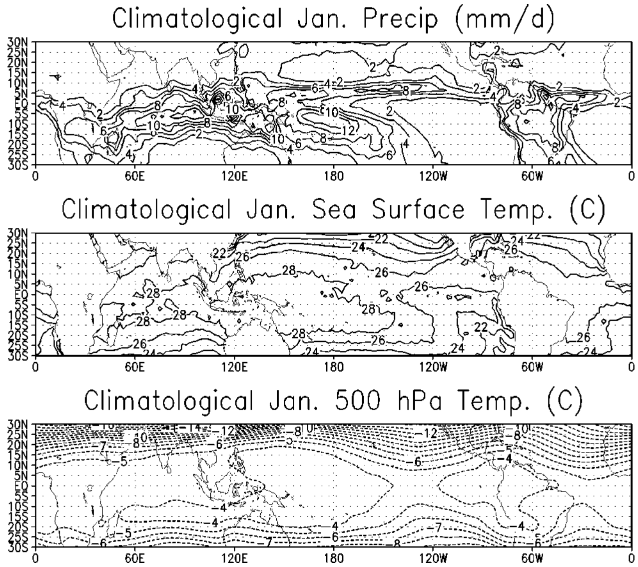

FIG. 1. Climatological January mean fields of precipitation (upper panel, mm d^{-1}), sea surface temperature (middle panel, °C), and temperature at 500 hPa (lower panel, °C).

is capable of balancing large Φ gradients (except on the short time scales characteristic of gravity waves). The Coriolis effect is not the strong constraint on the dynamics that it is at higher latitudes. Just as importantly, the character of the tropical circulation is strongly determined by the spatial and temporal structure of the heating Q_T, so that the assumption of dry adiabatic dynamics is inappropriate, even qualitatively. If we wish to define a balanced dynamics that is relevant to the large-scale tropical circulation, it needs to be based on a different set of assumptions from those used in the extratropics.

Figure 1 shows several climatological January-mean fields according to recent observational estimates. Figure 1 shows precipitation,[10] sea surface temperature (SST),[11] and atmospheric temperature[12] at a pressure of 500 hPa, in midtroposphere. Precipitation (and hence Q_c), is highly structured in space. The SST pattern is broader in scale, but upon examination bears a clear resemblance to the precipitation; a warmer ocean surface appears to support more rain. The midtropospheric temperature, however, is nearly constant throughout the entire tropical belt (notice that the contour interval for the bottom panel is half that for the middle one).

The smallness of the tropospheric temperature gradient follows from the smallness of $\nabla\Phi$ by hydrostatic balance, since given small surface pressure gradients (which apart from hurricanes one can assume are also small in the tropics, again because f is small) the geometric height of a pressure surface, which determines Φ, is itself determined by the vertical integral of temperature from the earth's surface up to the given pressure surface. Dynamically, the small geopotential gradients (equivalent to small pressure gradients in geometric coordinates) result from the process of geostrophic adjustment.[13] Given initial mass and flow fields which are not in geostrophic balance, gravity waves adjust them toward a state which is. On the f plane, this adjustment takes a time of order f^{-1}, and the balanced state eventually reached has characteristic length scale $L_R = c/f$, where c is the gravity wave speed; L_R is known as the Rossby deformation radius. At the equator, L_R becomes infinite, and the f-plane analysis predicts that balanced pressure and and temperature variations will have infinite spatial scale. This is qualitatively correct for the zonal variations, which have very little amplitude near the equator as Fig. 1 shows. The meridional structure requires a little more explanation than the f-plane analysis can provide. On the equatorial β-plane, we can heuristically derive the natural meridional adjustment length scale L_β by computing the latitude at which the distance from the equator is equal to the local L_R, using $f = \beta y$,

$$L_\beta = \frac{c}{\beta L_\beta},$$

and so arrive at $L_\beta = (c/\beta)^{1/2}$. This is of order 2000 km for motions spanning the depth of the troposphere in the vertical, and comes out as the natural meridional scale in both the linear equatorial wave problem[14] and nonlinear axisymmetric models of the overturning Hadley circulation in the tropics.[15,16]

Temperature and pressure variations can therefore be expected to (and do) have very little horizontal structure within the tropics, for the same reason that such variations are near geostrophic balance with the winds outside the tropics. When the adjustment is to a sustained convective heating rather than an initial unbalanced anomaly, the process is sometimes called "convective adjustment"[17] rather than geostrophic adjustment, but there is no fundamental difference.

Given the smallness of temperature gradients, and the *a priori* statement that the heating Q_T is important in determining the flow, it follows that the dominant free-tropospheric balance—that is, excluding the PBL—on large scales in Eq. (1) is

$$w \frac{\partial \theta}{\partial z} \approx Q_c + Q_R. \tag{6}$$

The validity of the approximate equality (6) has been long recognized, but its use as the dominant balance in a systematic scaling argument leading to a balanced dynamics is basically quite recent, and it is this that we emphasize here. Some earlier studies[18–21] formulated approximate reduced equations for dry dynamics—that is, Q_T given, and no moisture equation such as Eq. (2) included in the model—in which Eq. (6) was used as the temperature equation. Other studies have focused on the thermodynamics or "physics" which determine Q_c, Q_R, E, etc., in models which use Eq. (6) but do not solve real fluid dynamical, that is momentum, equations.[22–28] That these two aspects of the problem have often been studied separately is presumably a reflection of the complexity of the moist problem, in which the source and sink terms on the right-hand sides of Eqs. (1)–(4) are explicitly parametrized in terms of the large-scale state variables T, q, etc.

It has been recently argued[27,29,30] that the problem of large-scale tropical moist dynamics can be significantly simplified by the systematic use of Eq. (6), which we call here the *weak temperature gradient* (WTG) approximation. The essential idea is that for some applications, free-tropospheric variations of temperature in the horizontal and in time are small enough to be neglected *both* on the left-hand side of Eq. (1), *and* wherever they may appear in the parametrized physics determining the right-hand sides of Eqs. (1) and (2). Writing these dependencies out (partly) explicitly, in general we have

$$\frac{D\theta}{Dt}=Q_c(T,q,\ldots)+Q_R(T,q,\ldots), \tag{7}$$

$$\frac{dq}{dt}=Q_q(T,q,\ldots), \tag{8}$$

where the dependencies on the right-hand sides may be nonlocal in the vertical, so that for example the convective heating at one height generally depends on the entire vertical profile of T and q. Again we emphasize the dependence on T and q as these are the primary variables to which most modern schemes respond. We expect convection to be most sensitive to the lower- and mid-tropospheric humidity (as opposed to the radiative greenhouse effect, for which upper-tropospheric humidity is most important[3]). Under the WTG approximation Eqs. (7) and (8) become

$$w\frac{\partial\overline{\theta}}{\partial z}=Q_c(\overline{T},q,\ldots)+Q_R(\overline{T},q,\ldots), \tag{9}$$

$$\frac{dq}{dt}=Q_q(\overline{T},q,\ldots), \tag{10}$$

where the overbar represents a horizontal average so that $\overline{T}$, $\overline{\theta}$ are functions only of z, and perhaps time, though for many purposes we can take the tendency of the tropical mean temperature to be negligibly small. Now, given $\overline{T}(z)$, $q(x,y,z,t)$, and the other parameters on which Q_c and Q_R depend, using mass conservation and boundary conditions (9) determines the horizontally divergent part of the flow *independently* of the momentum equation. Equation (9) is a balance approximation, analogous to Eq. (5). As in quasigeostrophic theory, free inertia-gravity wave solutions are excluded when Eq. (9) is used in place of Eq. (1). Physically, this is no accident. Both approximations assume the geostrophic adjustment process to be instantaneous, the difference being only whether f is large or small.

Of Q_c and Q_R, it is the former which has the largest variability in space and time and thus whose spatial and temporal variations predominantly control the large-scale divergent flow through Eq. (6).

IV. APPLICATION I: SINGLE-COLUMN MODELING

The conceptually simplest application of WTG, and yet the most useful for understanding the basic idea and perhaps the most important in practice, is to single-column modeling. A single-column model (SCM) has a vertical dimension only. For our purposes such a model should be thought of as a single grid square of a global numerical model, and SCMs can be used as such to test the physical parametrizations of global models,[31] though they have had other distinct applications in climate theory (e.g., Refs. 32 and 33). Lacking a horizontal dimension, and given the assumption of hydrostatic balance in the vertical, a SCM has essentially no fluid dynamics as such, i.e., it cannot have a meaningful momentum equation. Nor does it have a closed mass budget in general. It must be told what the rest of the atmosphere around it is doing, and in practice this has been done by specifying either the ''large-scale'' vertical velocity, w, or the entire vertical advection terms, in Eqs. (1) and (2), as well either specifying or neglecting the horizontal advection terms. Because the dominant balance in the tropical free troposphere is Eq. (6), variations in Q_c are large compared to those in Q_R, and the static stability $\partial\theta/\partial z$ cannot change much due to the atmosphere's need to maintain approximate neutrality to moist convection,[32,34,35] this practice strongly constrains Q_c, and thus also the precipitation, since by conservation of energy the mass-weighted vertical integral of Q_c is just the net latent heating from condensation in the column (which is proportional to the precipitation if we neglect the storage of condensed water in the atmosphere, as we can to a first approximation). This prevents one from using a SCM to understand what controls tropical precipitation, independently of all the important issues involved in constructing the physical parameterizations.

Sobel and Bretherton[27] suggested enforcing Eq. (6) explicitly in a SCM, above a nominal boundary layer near the surface. This eliminates temperature as a prognostic variable in that region, so temperature must be specified there. The argument for this is that the temperature at a given vertical level in a limited horizontal area is constrained to be nearly the same as that at other locations in the tropics, by the large-scale nature of dynamical adjustment near the equator discussed previously. This temperature is thus determined by the tropical mean heat budget, which is insensitive to processes occurring in the limited area in question, assuming it is small in spatial extent compared to the rest of the global tropics (and that the spatial distribution of the terms in the heat budget is reasonably smooth so that there is not a singularity at the location of our SCM). The advantage to this approach is that w and hence Q_c and precipitation are now true output variables, determined by model physics (through the physics' determination of Q_T and Q_q interactively with a fully prognostic moisture equation) rather than being directly constrained by the inputs. A SCM run in this mode can be used to understand the influences that SST, surface wind speed and evaporation, radiative processes, and horizontal moisture advection can have on precipitation.

The influence of horizontal moisture advection is of particular interest here and is illustrated by Figs. 2 and 3, adapted from Ref. 27, to which the reader is referred for more details of the procedure than I present here. A particular single column model, developed by Rennó *et al.*,[36,37] was modified so that above a nominal boundary layer with top at 850 hPa (this model is formulated in pressure coordinates; pressure decreases upwards from a surface value slightly greater than 1000 hPa, with $1\,\text{hPa}=100\,\text{N}\,\text{m}^{-2}$), the temperature is fixed and the vertical velocity diagnosed from Eq. (6) as described previously. Q_T is determined using the convective parametrization of Emanuel[38] and the radiative parametrization of Chou *et al.*[39] Below 850 hPa, the tempera-

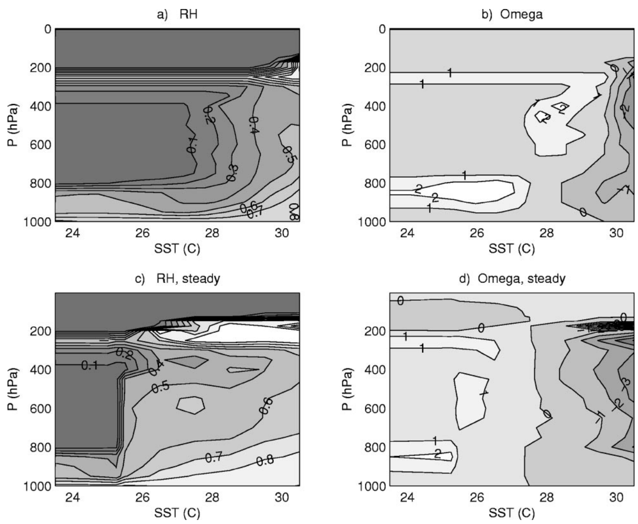

FIG. 2. Calculations done with a single column model in which the temperature is fixed and ω diagnosed from Eq. (6) above 850 hPa. Below 850 hPa temperature is interactive. Relative humidity (RH) and pressure vertical velocity (ω) are plotted. The top panels show a single time-dependent calculation in which the SST is increased in time, while the bottom panels show steady-state solutions reached for time-independent SST in the same range. Other parameters are identical in the two calculations. Further details are in the text.

ture is interactive, reflecting the fact that near the surface, turbulent fluxes originating at the surface (Q_{diff}) overwhelm the large-scale dynamical adjustment and tie the near-surface air temperature to the local underlying surface temperature.

The upper panels of Fig. 2 show relative humidity (RH) and pressure vertical velocity, $\omega = dp/dt$ (which is *negative* for upward motion), as a function of SST for a time-dependent simulation in which the SST is increased in time at a rate comparable to that which would be experienced as a lower boundary condition by a column of air traveling equatorward with the eastern Pacific trade winds, starting in the subtropics. The initial condition is the steady-state solution the model obtains, with no horizontal advection, for the initial SST of 23.5 °C. (This initial condition is different than that used in the corresponding time-dependent calculation shown by Sobel and Bretherton.[27]) This initial condition has near-zero RH above 850 hPa, because downward motion dries the atmosphere and without horizontal advection there

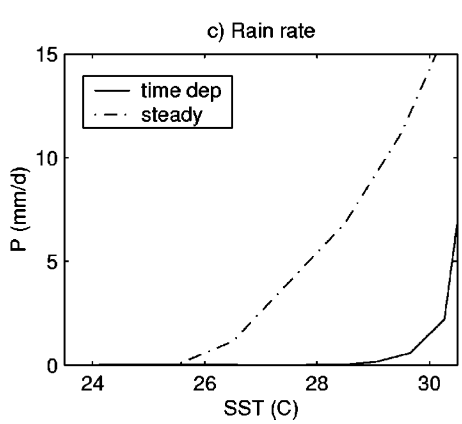

FIG. 3. Precipitation for the same set of calculations as in Fig. 2.

is no source to balance this drying. The lower panels show the corresponding steady-state solutions for each value of SST corresponding to those traversed by the time-dependent calculation. In the time-dependent calculation, the low values of RH persist to much higher SST than in the steady-state calculations. This can be viewed as the effect of horizontal advection in a Lagrangian reference frame, assuming the entire column moves with a uniform velocity (no vertical shear). This lower RH in the time-dependent calculation *causes* the onset of precipitation (Fig. 3) and upward motion [Figs. 2(b) and 2(d)] to be delayed to considerably higher SST than in the steady calculations. The casual role of the humidity field can be inferred from the fact that nothing else to which the convective and radiative parametrizations are sensitive differs in the two calculations (except for a small difference the PBL temperature which can be shown to have a secondary influence). The free tropospheric temperature, surface wind speed, and other parameters are all identical, but at any given SST the time-dependent calculation has a ''memory'' of the drier conditions corresponding to the lower SST upstream.

The vertical structure of the RH deserves some consideration. Deep convection (here, parametrized) is certainly sensitive to boundary-layer humidity, since this directly influences the buoyancy of rising parcels. Boundary layer RH differences between the two calculations are small but not insignificantly so. The free tropospheric RH, which shows much greater differences between the simulations because it adjusts more slowly to surface conditions (and does not adjust at all if there is no convection to communicate those conditions to the free troposphere), also affects deep convection, because entrainment of environmental air into cloud updrafts reduces their buoyancy by an amount that increases as the environment gets drier. This mechanism, long known,[40] has gained attention recently,[26,41,42] and some interesting variations on it have been proposed.[43,44] Its relative importance in this model could be ascertained through sensitivity studies, but certainly depends on the convective scheme's entrainment parametrization and this, in any scheme, is highly uncertain at present. The relative importance of boundary layer and free-tropospheric humidity in controlling convection therefore remains a fairly open issue.

V. THE WTG APPROXIMATION IN THE QUASIEQUILIBRIUM TROPICAL CIRCULATION MODEL EQUATIONS

A. Equations

To arrive at a simple model of moist tropical dynamics which can illustrate the moisture/flow interaction in a more complete way, we consider the WTG approximation as applied to models with highly truncated vertical structure but full horizontal structure. The most well-developed example is the ''quasiequilibrium tropical circulation model'' (QTCM).[45,46] Thus we assume single, fixed vertical basis functions for temperature, humidity, and both horizontal and vertical velocity, reducing the remaining dynamics to essentially two horizontal dimensions. The basis functions in the QTCM are ''tailored'' to give empirically good solutions in the tropics and (relatedly) to be consistent with physical assumptions in the convective parametrization, and consistent with each other using basic constraints such as mass and energy conservation and hydrostatic balance.[45] The moisture basis function is heavily weighted toward the lower troposphere, so the appropriate measure of instability to convection includes the effects of both PBL and free tropospheric moisture.

The reader is referred to the original papers for details of the QTCM formulation, which we follow here except that some notation is streamlined and, since we do not attempt to compute solutions for realistic boundary conditions, we can avoid being very specific about the physical parametrizations. Reduction of the three-dimensional equations to a shallow-water system can also be done by other means, and is in fact familiar in tropical meteorology. An advantage to this particular equation set is that it has been coded in an explicit climate model with a consistent energy budget and a specific set of physical parametrizations—which are simple but comparable in construction to those in more sophisticated models—so that solutions exist which can be compared directly with observations.[46]

Here, we discuss the following set in which the WTG approximation has been made, meaning that we explicitly drop all terms depending on horizontal temperature variations. We write the equations in vorticity-divergence form since this captures the causality of the WTG system best. We obtain for the simplest ''WTG QTCM'' system:

$$\overline{M}_S\delta = P(\overline{T},q) + Q_R(\overline{T},q,P), \tag{11}$$

$$\frac{\partial \zeta}{\partial t} + \mathbf{v}\cdot\nabla(f+d\zeta) + (d\zeta+f)\delta = -\alpha\zeta, \tag{12}$$

$$\hat{b}\left(\frac{\partial q}{\partial t} + \mathbf{v}\cdot\nabla q\right) - M_q\delta = E(T_S,q,\mathbf{v}) - P(\overline{T},q). \tag{13}$$

Here T is the heat capacity of air times the temperature, q the latent heat of vaporization of water times specific humidity (so T and q have the same units of J/kg^{-1}, which is often convenient), and $\mathbf{v}$ the horizontal velocity, each function of x, y, t implicitly multiplying a basis function in the vertical coordinate. The basis functions for T and q have a single sign in the troposphere, while that for $\mathbf{v}$ changes sign once with height, the sign in the above mentioned equations representing its value in the upper troposphere. The constant coefficients d and $\hat{b}$ arise from projection on the basis functions. We have defined

$$\delta = \nabla\cdot\mathbf{v},$$

$$\zeta = \nabla\times\mathbf{v},$$

and T_S is the surface temperature (assumed given). Here we write P in place of Q_c because all quantities can now be viewed as vertically averaged in such a way that, since the temperature and moisture variables now have the same units, precipitation and convective heating are one and the same. Additionally, horizontal diffusion terms have been neglected, and in Eq. (12), vertical advection of momentum has been neglected. Neglect of these effects is for simplicity only, and is not necessary for the theory. M_S is the appropriate dry static stability, proportional to $\partial\theta/\partial z$, while M_q is the ''gross moisture stratification,'' proportional to $\partial q/\partial z$ and in this system a linearly increasing function of q. ∇ in the above represents the horizontal gradient only, and α is a constant Rayleigh drag coefficient. Notice that the vertical velocity does not appear explicitly in the above-mentioned equations; by the assumption of a unique vertical structure it can be replaced by δ because the two are uniquely related by mass conservation.

B. Properties

The key approximation occurs in the temperature equation, here Eq. (11), in which tendency and horizontal advection terms have been neglected as in Eq. (6). Equation (11) can be viewed as a Poisson equation for a divergent flow potential χ, defined by $\nabla^2\chi = \delta$, and so given boundary conditions this equation yields the divergent part of the flow, independently from the momentum budget. The divergence δ thus obtained from Eq. (11) drives the vorticity equation (12) through the last term on the left-hand side of the latter. After solving another Poisson equation to obtain the stream function (ψ, with $\nabla^2\psi = \zeta$) and thus the rotational component of $\mathbf{v}$ from ζ, the entire flow is known and can be used in the advection terms for moisture and vorticity on the next time step. The moisture equation couples to the rest of the dynamics by the q dependence of P, Q_R, E.

The divergence equation (produced by taking the divergence of the momentum equation; not shown) is to a first approximation not needed at all, but becomes a diagnostic equation that can be used to estimate the small temperature perturbation that has been neglected in Eq. (11). The temperature in this system thus plays a role analogous to that of vertical velocity in quasi-geostrophy, a next-order correction that can be inferred by a consistency requirement on the leading-order solution. This requirement shows that the neglect of temperature variations on the left-hand side of Eq. (11) is formally valid for length scales small compared to L_R, and time scales large compared to f^{-f}, on an f plane.[29] On the equatorial β plane this needs some refinement, but at least for the steady linear problem the approximation does not seem to induce qualitative errors.[19,47] Validity of the neglect of temperature perturbations in evaluating P and R has been addressed to some degree in a few studies.[27,48,49] Several different versions of the dry WTG approximation, valid on different space and time scales, have been recently derived by Majda and Klein.[50]

Dynamically, the system above is the shallow water equations, but with the divergence determined by an additional scalar variable (q) with its own conservation equation, rather than by the interaction of the momentum equation with a free-surface mass conservation equation. The elimination of the latter interaction is what removes gravity waves (and, on the equatorial β plane, Kelvin and mixed Rossby-gravity waves) from the system. The system contains two scalars, vorticity and moisture, which are advected by the flow and which in turn feed back on the flow, as opposed to the one, PV, in quasigeostrophic dynamics. (Under WTG, because the temperature and hence stratification are horizontally constant at leading order, PV contains no information beyond what vorticity does. As a result there is no point in using PV per se within WTG and so we phrase our arguments in terms of vorticity.) In this and other respects, our system bears a strong formal similarity to the "zero Mach number equations"[51,52] of combustion theory. Both arise as asymptotic limits which, in the adiabatic limit, would yield nondivergent flow (three-dimensional in the zero Mach number case, two-dimensional in the WTG case[53]). In both cases the retention of the source terms at leading order instead yields an equation in which the source terms directly yield the divergence, which in turn drives a vorticity equation, yielding the total flow which advects constituents that are dynamically active through their effect on the source terms (moisture here, combustible species in the combustion case).

Here, neither vorticity nor moisture is generally well conserved. ζ is modified by both vortex stretching and friction, while q is strongly affected by vertical motion (expressed here by δ), as well as E and P. Nonetheless the horizontal advection terms in Eqs. (12) and (13) can be important, as illustrated above in the SCM results.

C. Behavior

The dry WTG shallow-water system, that is, a system equivalent to that given previously but with a given right-hand side to Eq. (11) and no moisture equation, appears to provide a fairly good approximation to the steady linear ("Gill"[54]) problem on the equatorial β plane[19,47] (that is, $f_0=0$, $f=\beta y$), which at first glance is surprising since the Kelvin wave, eliminated by WTG, is often thought to be an important component of the solution to that problem. Idealized shallow-water models of the nonlinear axisymmetric Hadley circulation[55] seem also to survive the WTG approximation with their key properties intact.

These dry solutions just show that the WTG approximation does not do major damage to previously well-understood, classic results in tropical dynamics. Interesting new behavior is more apparent when we consider the system with an interactive moisture equation. A first example of this is the linear wave modes which arise when the background state has a constant latitudinal moisture gradient.[29] Independent evidence for similar or related modes has been seen in three-dimensional, nonlinear numerical simulations of intraseasonal tropical variability.[56,57] These modes result from interactions between the moisture field, divergence, and rotational flow, and owe their existence to the horizontal moisture advection term.

Linearizing Eqs. (11)–(13) on the "midlatitude" β plane (the equatorial β plane would be more appropriate, but makes the analysis more complex—by adding a nonconstant coefficient—without fundamentally altering the essential wave dynamics) $f=f_0+\beta y$, and assuming a basic state moisture gradient $\partial\overline{q}/\partial y=\beta_q$ but no basic state flow, we obtain

$$\overline{M}_S\delta' = \frac{q'}{\tau}, \tag{14}$$

$$\frac{\partial \zeta'}{\partial t} + v'\beta + f_0\delta' = -\alpha\zeta', \tag{15}$$

$$\hat{b}\left(\frac{\partial q'}{\partial t} + v'\beta_q\right) - \overline{M}_q\delta' = -\frac{q'}{\tau}. \tag{16}$$

τ is a "convective time scale," typically hours to days but essentially a free parameter for our purposes. We have neglected perturbations in E and R for simplicity. These can be quite important in generating tropical atmospheric variability, but have been studied to a significant extent in other contexts[58–62] and we wish to focus on another process here.

Assuming solutions proportional to $\exp[i(\omega t-kx-ly)]$, we obtain a dispersion relation whose low frequency root can be approximated by

$$\omega = \frac{k(f_0 - L_q\beta)}{L_qK^2 + i(l + A^{-1}\beta k)}. \tag{17}$$

In this $K^2=k^2+l^2$ and $L_q=\overline{M}\beta_q$, where $\overline{M}=\overline{M}_S-\overline{M}_q$ is the mean gross moist stability,[22,45] is a length scale associated with the moisture gradient, and the constant $A=\beta_q/\tau\overline{M}_S\hat{b}$. These waves arise from interaction between the moisture, vorticity, and divergence fields. A moisture anomaly creates a precipitation and hence a divergence anomaly, which over a time scale f_0^{-1} spins up a vorticity anomaly; both the rotational and divergent flow anomalies advect the mean moisture and (planetary) vorticity gradients to propagate (and possibly damp or amplify) the moisture and vorticity anomalies.

The coupling between the two active scalars can be seen by considering, first, the uncoupled "dry" case $q'=0$, where the dispersion relation is that of the barotropic Rossby wave

$$\omega = \frac{-\beta k}{K^2}, \tag{18}$$

and then the coupled moist case, but on the f-plane $\beta=0$, where Rossby waves do not occur and for low frequencies we have

$$\omega = \frac{f_0 k}{L_qK^2 + il}. \tag{19}$$

Equations (18) and (19) are quite similar in form. Equation (19) allows growth and decay whereas Eq. (18) characterizes only neutral modes, but for $l=0$ the expressions are the same up to constants. For moisture decreasing poleward in the basic state, the moisture wave propagates eastward where the Rossby (vorticity) wave propagates westward. In the more general case where Eq. (17) holds, the propagation dynamics of the vorticity and moisture anomalies fight against each other and we get cancellation between the two terms in the numerator. Estimating the degree of cancellation from parameters consistent with observations is not entirely trivial, but it appears it is significant, to the degree that the sign as well as magnitude of the phase speed is indeterminate. To the extent there are unstable modes, the cancellation will reduce their growth rates. The instabilities (due only to the moisture wave dynamics) are at low wave numbers in any case, and the β-effect may well push them to scales larger than the circumference of the earth, so that no instability will actually be present although a weakly damped mode could still be excited by other processes.

The nonlinear behavior of the above system has so far only been studied in the very simplified context of the "Walker circulation," in which $f=0$, the domain is symmetric in y and therefore the problem is one-dimensional, and steady solutions are sought, with the circulation driven by spatial structure in the lower boundary condition (in the simplest case, SST) which influences E.[49] A particularly important nonlinearity, besides the advective ones, results from the fact that P cannot be negative. A key property of the solution is the size of the convective region, defined as the region where $P>0$. Although the horizontal moisture advection term is generally smaller in magnitude than the dominant terms in the moisture budget (E, vertical advection or $M_q\delta$, and P), it nonetheless plays an important role in determining the size of the convective region, reducing it significantly over what the local energetics would by themselves determine by bringing dry air into regions of high SST and suppressing rainfall there as seen in Figs. 2 and 3.

Additional nonlinear behavior is expected to result for the two-dimensional, equatorial beta-plane case $f=\beta y$, when the lower boundary condition has two-dimensional structure (as in observations) and transients are considered.

VI. CONCLUDING REMARKS

One can derive balanced dynamical equation systems for tropical atmospheric dynamics, based on the WTG approximation and including explicit moisture equations and convective parametrizations. Water vapor is dynamically active in any case—whether temperature variations are retained or not—but the WTG approximation brings it to the fore by eliminating other effects (gravity waves, baroclinic instability[29]). In our balanced tropical system there are two scalars, vorticity and moisture, which are advected and which influence the flow, as opposed to one, the potential vorticity, in extratropical, near-adiabatic balance models such as quasigeostrophy. The WTG system has a particularly close mathematical analogy with the zero Mach number equations of combustion theory.[51,52]

The physical and mathematical ramifications of the ideas presented previously have only begun to be worked out. In our examples either the vertical or horizontal dimension is removed, and not all important physical effects are included. Implementation of WTG in three spatial dimensions, with "full physics," presents a range of interesting problems. There are fundamental ones, such as determining the conditions of validity of the WTG approximation (especially when moist physics is explicitly included) and developing higher-order extensions. There are also practical ones, such as solving the equations (analytically or numerically) for realistic or idealized boundary conditions and forcings and comparing to observations or more exact models. Since there are several fundamental nonlinearities in the system, a rich spectrum of behavior is expected, and the results should have some relevance to practical questions in the major and growing field of global climate studies.

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[1] B. J. Hoskins, M. E. McIntyre, and A. W. Robertson, Q. J. R. Meteorol. Soc. **111**, 877–946 (1985).
[2] J. G. Charney, Geofys. Pub. **17**, 17 (1948).
[3] I. M. Held and B. J. Soden, Ann. Rev. Energy Environ. **25**, 441–475 (2000).
[4] My presentation of the equations follows D. G. Andrews, J. R. Holton, and C. B. Leovy, *Middle Atmosphere Dynamics* (Academic, New York, 1987), 487 pp.
[5] M. Yanai, S. Esbensen, and J.-H. Chu, J. Atmos. Sci. **30**, 611–627 (1973).
[6] See A. Arakawa, "Closure assumptions in the cumulus parametrization problem," in *The Representation of Cumulus Convection in Numerical Models*, edited by K. A. Emanuel and D. J. Raymond (American Meteorological Society, Boston, 1993), 246 pp. for a basic framing of the convective parametrization problem. Other papers in the same volume provide descriptions of many specific parametrizations that are or have been widely used in numerical models of the atmosphere.
[7] K. A. Emanuel, *Atmospheric Convection* (Oxford University Press, Oxford, 1994), 580 pp.
[8] *The Representation of Cumulus Convection in Numerical Models*, edited by K. A. Emanuel and D. J. Raymond (American Meteorological Society, Boston, 1993), 246 pp.
[9] R. K. Smith, Ed., *The Physics and Parameterization of Moist Atmospheric Convection* (Kluwer, Dordrecht, 1997).
[10] P. Xie and P. A. Arkin, Bull. Am. Meteorol. Soc. **78**, 2539–2558 (1997).
[11] R. W. Reynolds and T. M. Smith, J. Clim. **7**, 929–948 (1994).
[12] E. Kalnay *et al.*, Bull. Am. Meteorol. Soc. **77**, 437–471 (1996).
[13] For example, see A. E. Gill, *Atmosphere-Ocean Dynamics* (Academic, New York, 1982), pp. 191–203.
[14] T. Matsuno, J. Met. Soc. Japan **44**, 25–42 (1966).
[15] E. K. Schneider, J. Atmos. Sci. **34**, 280–296 (1977).
[16] I. M. Held and A. Y. Hou, J. Atmos. Sci. **37**, 515–533 (1980).
[17] C. S. Bretherton and P. K. Smolarkiewicz, J. Atmos. Sci. **46**, 740–759 (1989).
[18] I. M. Held and B. J. Hoskins, Adv. Geophys. **28A**, 3–31 (1985).
[19] J. D. Neelin, Q. J. R. Meteorol. Soc. **114**, 747–770 (1988).
[20] G. L. Browning and H.-O. Kreiss, J. Atmos. Sci. **54**, 1166–1184 (1997).
[21] G. L. Browning, H.-O. Kreiss, and W. H. Schubert, J. Atmos. Sci. **57**, 4008–4019 (2000).
[22] J. D. Neelin and I. M. Held, Mon. Weather Rev. **115**, 3–12 (1987).
[23] R. T. Pierrehumbert, J. Atmos. Sci. **52**, 1784–1806 (1995).
[24] R. L. Miller, J. Clim. **10**, 409–440 (1997).
[25] K. Larson, D. L. Hartmann, and S. A. Klein, J. Clim. **12**, 2359–2374 (1999).
[26] D. J. Raymond, Q. J. R. Meteorol. Soc. **564**, 889–898 (2000).
[27] A. H. Sobel and C. S. Bretherton, J. Clim. **13**, 4378–4392 (2000).
[28] J. C. H. Chiang and A. H. Sobel, "Tropical tropospheric temperature variations caused by ENSO and their influence on the remote tropical climate" J. Clim. (in press).
[29] A. H. Sobel, J. Nilsson, and L. M. Polvani, J. Atmos. Sci. **58**, 3650–3665 (2001).
[30] D. J. Raymond and A. H. Sobel, "Forecasting convection, precipitation, and vertical motion in the tropics," Proceedings of the International Work-

shop on the Dynamics and Forecasting of Tropical Weather Systems, 2001.
[31] D. A. Randall, K.-M. Xu, R. J. C. Somerville, and S. Iacobellis, J. Clim. **9**, 1683–1697 (1994).
[32] S. Manabe and R. F. Strickler, J. Atmos. Sci. **21**, 361–385 (1964).
[33] S. Manabe and R. T. Wetherald, J. Atmos. Sci. **24**, 241–259 (1967).
[34] A. Arakawa and W. H. Schubert, J. Atmos. Sci. **31**, 674–701 (1974).
[35] K. A. Emanuel, J. D. Neelin, and C. S. Bretherton, Q. J. R. Meteorol. Soc. **120**, 1111–1143 (1994).
[36] N. O. Rennó, K. A. Emanuel, and P. H. Stone, J. Geophys. Res., [Atmos.] **99**, 14429–14441 (1994).
[37] N. O. Rennó, P. H. Stone, and K. A. Emanuel, J. Geophys. Res., [Atmos.] **99**, 17001–17020 (1994).
[38] K. A. Emanuel, J. Atmos. Sci. **48**, 2313–2335 (1991).
[39] M.-D. Chou, D. P. Kratz, and W. Ridgway, J. Clim. **4**, 424–437 (1991).
[40] H. Stommel, J. Atmos. Sci. **8**, 127–129 (1951).
[41] S. C. Sherwood, Mon. Weather Rev. **127**, 2977–2991 (1999).
[42] C. Lucas, E. J. Zipser, and B. S. Ferrier, J. Atmos. Sci. **57**, 2351–2373 (2000).
[43] D. E. Kingsmill and R. A. Houze, Jr., Q. J. R. Meteorol. Soc. **125**, 1165–1207 (1999).
[44] D. E. Kingsmill and R. A. Houze, Jr., Q. J. R. Meteorol. Soc. **125**, 1209–1229 (1999).
[45] J. D. Neelin and N. Zeng, J. Atmos. Sci. **57**, 1741–1766 (2000).
[46] N. Zeng, J. D. Neelin, and C. Chou, J. Atmos. Sci. **57**, 1767–1796 (2000).
[47] C. S. Bretherton and A. H. Sobel, ''The Gill model and the weak temperature gradient (WTG) approximation,'' J. Atmos. Sci. (to be published).
[48] H. Su and J. D. Neelin, ''Teleconnection mechanisms for tropical Pacific descent anomalies during El Niño,'' J. Atmos. Sci. (submitted).
[49] C. S. Bretherton and A. H. Sobel, ''A simple model of a convectively coupled Walker circulation using the weak temperature gradient approximation,'' J. Clim. (to be published).
[50] A. J. Majda and R. Klein, ''Systematic multi-scale models for the tropics,'' J. Atmos. Sci. (submitted).
[51] A. J. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables* (Springer, New York, 1984), 159 pp.
[52] A. J. Majda and J. Sethian, Combust. Sci. Technol. **42**, 185–205 (1985).
[53] J. G. Charney, J. Atmos. Sci. **20**, 607–609 (1963).
[54] A. E. Gill, Q. J. R. Meteorol. Soc. **106**, 447–462 (1980).
[55] L. M. Polvani and A. H. Sobel, ''The Hadley circulation and the weak temperature gradient approximation,'' J. Atmos. Sci. **59**, 1744–1752 (2002).
[56] D. J. Raymond, J. Atmos. Sci. **58**, 2807–2819 (2001).
[57] S.-P. Xie and N. Saiki, J. Meteor. Soc. Jpn. **77**, 949–968 (1999).
[58] K. A. Emanuel, J. Atmos. Sci. **44**, 2324–2340 (1987).
[59] J. D. Neelin, I. M. Held, and K. H. Cook, J. Atmos. Sci. **44**, 2341–2348 (1987).
[60] K. A. Emanuel, J. Atmos. Sci. **50**, 1763–1775 (1993).
[61] D. J. Raymond and D. J. Torres, J. Atmos. Sci. **55**, 1771–1790 (1998).
[62] D. J. Raymond, J. Atmos. Sci. **57**, 1286–1297 (2000).