Cross-channel advective–diffusive transport by a monochromatic traveling wave

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The cross-channel tracer flux due to the combined effects of advection and diffusion is considered for two-dimensional incompressible flow in a channel, where the flow is that due to a monochromatic traveling wave and the boundary conditions at the walls are fixed tracer concentration. The tracer flux is computed numerically over a wide range of the parameters $\epsilon = U/c$ and $\delta = K/cL$, with U the maximum fluid velocity, c the wave phase speed, K the tracer diffusivity, and L the channel width. Prior work has used analytical methods to obtain solutions for δ either infinite (stationary overturning cells) or small. In addition to the full numerical solutions, solutions obtained using mean field theory are presented, as well as a new asymptotic solution for small ϵ, and one for small δ due to Flierl and Dewar. The various approximations are compared with each other and with the numerical solutions, and the domain of validity of each is shown. Mean field theory is fairly accurate compared to the full numerical solutions for small δ, but tends to underpredict the tracer flux by 30–50% for larger δ. The asymptotic solution derived by Flierl and Dewar for small δ is found to break down when $\delta \sim \epsilon^{-2}$ rather than when $\delta \sim 1$ as suggested by the original derivation, and a scaling argument is presented which explains this. © *2000 American Institute of Physics.* [S1070-6631(00)00106-9]

I. INTRODUCTION

Simple, idealized flows are an attractive context for understanding basic aspects of passive tracer transport. The insights gained from studying such models provide the building blocks for analyzing and interpreting more realistic flows. Here we present numerical calculations of the cross-channel advective-diffusive transport by a single traveling sinusoidal wave. Many examples of wavelike flows can be found, for example, in the oceans and atmosphere, although real flows are neither purely monochromatic nor truly steady in any reference frame. In temporally varying flows, chaotic advection is possible, in contrast to the present case. It is not clear how strong chaotic advection needs to be in order to invalidate the present results, and this question is not addressed here.

The flow we shall use is two-dimensional and incompressible, so that the advection-diffusion equation can be stated in terms of a stream function, ψ,

$$\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial q}{\partial x} = K\nabla^2 q, \tag{1}$$

where q is the concentration of tracer and K the diffusivity. The channel is periodic in x with length $2L$, and has rigid walls at $y=0,L$. The boundary conditions on q are

$$q(x,0)=0; \quad q(x,L)=\Delta q. \tag{2}$$

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We study the flow

$$\psi = \psi_0 \sin(\pi y/L)\sin(\pi[x-ct]/L). \tag{3}$$

The quantity of primary interest in the solution is the nondimensional cross-channel tracer flux, or Nusselt number $\mathrm{Nu} = -FL/K\Delta q$, where

$$F = \overline{\frac{\partial \psi}{\partial x}q} - K\frac{\partial \overline{q}}{\partial y},$$

and an overbar represents spatial averaging in the x direction over a wavelength.

Advective-diffusive transport in similar flows has been studied by others, including Shraiman,[1] Rosenbluth *et al.*,[2] and Knobloch and Merryfield.[3] These considered infinite arrays of circulation cells, rather than flow in a channel spanned by a single cell with fixed tracer concentration boundary conditions as here, though the two are essentially isomorphic if the array of cells is aligned parallel to the channel walls. More significantly, Shraiman and Rosenbluth *et al.* considered only the case $c=0$, while Knobloch and Merryfield considered (among other flows) a traveling wave, but computed only the transport in the longitudinal (x) direction. For the case $c=0$ the transport in the two directions is governed by essentially the same physics, apart from straightforward aspect ratio effects. However, when $c\neq 0$ the two directions are fundamentally different due to the presence of a Stokes' drift in x but not y. Flierl and Dewar[4] considered cross-channel transport under fixed q boundary conditions by a general traveling flow disturbance, but examined the asymptotic behavior in the limit where the wave period is short compared to the diffusion time scale.

Following Flierl and Dewar, we nondimensionalize (1), consider it in a reference frame moving at speed c, and assume steady state to obtain

$$\frac{\partial \Psi}{\partial x}\frac{\partial q}{\partial y}-\frac{\partial \Psi}{\partial y}\frac{\partial q}{\partial x}=\delta\nabla^2 q, \tag{4}$$

where $\Psi=\epsilon\psi+y$, $\psi=(1/\pi)\sin(\pi y)\sin(\pi x)$, $\delta=K/cL$, $\epsilon=U/c$. For the present flow, the velocity scale is $U=\psi_0\pi/L$. The boundary conditions are $q(x,0)=0$ and $q(x,1)=1$. Flierl and Dewar considered the limit $\delta\ll 1$. When $c=0$, by contrast, δ is infinite. We are not aware of any previous calculations of cross-channel transport when δ is neither very small nor very large. In this case asymptotic theoretical approaches are inapplicable and numerical solutions are required.

We present numerical calculations of cross-channel transport for the problem described above, covering a range of phase speeds c corresponding to small, large, and intermediate δ. In addition to solving the full problem, we also present numerical calculations using the mean-field approximation. We perform calculations for a range of Peclet numbers $\mathrm{Pe}=\psi_0/K=\epsilon/\delta$ with a maximum $\mathrm{Pe}=1000$. The Flierl and Dewar solution is computed for comparison with the numerical results. New asymptotic solutions are also presented which are valid for small ϵ.

By comparison with the numerical results and with each other, the various approximate solutions are evaluated and the domain of validity of each is shown. Flierl and Dewar's asymptotic solution breaks down when $\delta\sim\epsilon^{-2}$ or greater. A scaling argument is presented which explains this. The mean-field approximation reproduces the numerical results fairly well for a large range in δ, but produces an error in the nondimensional tracer flux Nu which can be as large as 50% as $\delta\rightarrow\infty$.

II. REVIEW OF FLIERL AND DEWAR'S RESULTS

Flierl and Dewar considered not just (3), but all periodic flows

$$\psi=\psi(x-ct,y).$$

In the limit where the RHS of (4) can be neglected (which Flierl and Dewar took, plausibly, to be the case when $\delta\ll 1$; but see Sec. IV B 2) the lowest order solution is

$$q=q(\psi) \tag{5}$$

so that the tracer concentration is uniform along streamlines. Additionally, an integral constraint (true for all δ) requires that the net flux of tracer through any closed streamlines which do not intersect the boundaries must vanish; this implies that the tracer concentration must be uniform within such streamlines (see also Rhines and Young[5]). This constraint, combined with (5), allows Nu to be determined

$$\mathrm{Nu}=\left\{\int_0^1 d\Psi\left[\frac{\gamma}{\int_{\Psi=\Psi_0}|\nabla\Psi|\,dl}\right]\right\}^{-1}, \tag{6}$$

where dl increments distance along a streamline in the comoving frame. γ is the ratio of the spacing between eddies to the width of the domain, which is unity for the case of (3). Note that since δ does not appear in (6), in this case Nu is independent of the diffusivity, and hence the dimensional flux across the channel is proportional to K.

Flierl and Dewar did not explicitly consider the case of a stationary ($\delta=\infty$) periodic flow, but they did discuss an axisymmetric swirling flow in a circular domain, with the tracer concentration varying sinusoidally with distance along the circular boundary. This is fundamentally similar to the case of a stationary periodic cellular flow, and, consistently with results in such flows,[1,2,3] the appropriately defined Nu is proportional to the square root of the appropriately defined Pe for $\mathrm{Pe}\gg 1$. In dimensional terms, the flux across the domain is proportional to the square root of K, in contrast to the small δ case.

III. FURTHER ANALYTICAL RESULTS

We can make some analytical progress by considering the equations for the channel mean value $\overline{q}(y)$ and the fluctuations $q'(x,y,t)$. The mean is altered by diffusion and eddy fluxes

$$\frac{\partial\overline{q}}{\partial t}+\epsilon\frac{\partial}{\partial y}\overline{v'q'}=\delta\frac{\partial^2}{\partial y^2}\overline{q}. \tag{7}$$

The fluctuations in the scalar are produced by advection of the mean gradients; in the moving reference frame,

$$\frac{\partial q'}{\partial t}-\frac{\partial q'}{\partial x}+\epsilon v'\frac{\partial\overline{q}}{\partial y}+\epsilon\left[\frac{\partial}{\partial x}(u'q')+\frac{\partial}{\partial y}(v'q'-\overline{v'q'})\right]$$
$$=\delta\nabla^2 q'. \tag{8}$$

In these equations, $v'=\psi_x=\cos(\pi x)\sin(\pi y)$ and $u'=-\psi_y=-\sin(\pi x)\cos(\pi y)$.

A. Weak waves

When the flows are weak so that ϵ is small (compared to both 1 and δ, implying $\mathrm{Pe}\ll 1$), we see from (8) that $q'\sim\epsilon$ and, from (7), that the mean tracer value is $\overline{q}=y+O(\epsilon^2)$. Therefore our first estimate of the fluctuations comes from solving

$$\frac{\partial q'}{\partial x}+\delta\nabla^2 q'=\epsilon v'=\frac{\epsilon}{2}e^{i\pi x}\sin(\pi y)+\text{c.c.}$$

giving

$$q'=\frac{\epsilon}{2}\frac{1}{i\pi-2\delta\pi^2}e^{i\pi x}\sin(\pi y)+\text{c.c.}$$
$$=-\frac{\epsilon}{2}\left(\frac{\frac{i}{\pi}+2\delta}{1+4\delta^2\pi^2}\right)e^{i\pi x}\sin(\pi y)+\text{c.c.}$$

The eddy transport is

$$\overline{v'q'}=-\frac{\epsilon\delta}{1+4\delta^2\pi^2}\sin^2(\pi y).$$

If we now consider the steady version of the mean Eq. (7), we can integrate once to find

$$\overline{\epsilon v' q'} = \delta \frac{\partial \overline{q}}{\partial y} - \delta \, \mathrm{Nu}.$$

A second vertical integral gives

$$\mathrm{Nu} = 1 - \frac{\epsilon}{\delta} \int_0^1 dy \overline{v' q'}$$

(which is a general statement, not limited to weak waves). If we do substitute the weak wave estimate of the flux (equivalent to solving for the corrections to $\overline{q}$), we find

$$\mathrm{Nu} = 1 + \frac{\epsilon^2}{2} \left(\frac{1}{1 + 4\delta^2 \pi^2} \right)$$

as our estimate of the Nusselt number. If ϵ and δ are both small compared to 1, but $\epsilon \gg \delta$ so that $\mathrm{Pe} \gg 1$ (as appropriate for comparison to the numerical results below), a slight modification of the above analysis yields

$$\mathrm{Nu} = 1 + \frac{\epsilon^2}{2}.$$

B. Mean-field equations

The mean-field equations (cf., Herring[6]) assume that the dominant flux comes from the perturbations of the same scale as the wave. In effect, we drop the terms in the square brackets in (8) so that the fluctuation equation loses its ''statistical nonlinearity.''[7] The mean-field system as a whole retains this nonlinearity, however, because the structure of $\overline{q}$ in the y direction is altered by the fluctuations whose own y structure, in turn, depends on $\partial \overline{q} / \partial y$. If we take

$$q' = \hat{q}(y) e^{i\pi x} + \mathrm{c.c.}$$

we must then solve the coupled problem

$$\left[i\pi + \delta \frac{\partial^2}{\partial y^2} - \delta^2 \pi^2 \right] \hat{q} = \frac{\epsilon}{2} \sin(\pi y) \frac{\partial \overline{q}}{\partial y},$$

$$\frac{\epsilon}{2} \sin(\pi y)(\hat{q} + \hat{q}^*) = \delta \frac{\partial \overline{q}}{\partial y} - \delta \, \mathrm{Nu}$$

with $\overline{q}(0) = 0$, $\overline{q}(1) = 1$, $\hat{q}(0) = \hat{q}(1) = 0$. We solve this in spectral space in y with the same procedures used in the full two-dimensional model, as described next.

IV. NUMERICAL CALCULATIONS

A. Code description

Our numerical code for solving (1) uses the spectral method with a Fourier basis. The algorithm is fully spectral, not pseudospectral; the advective terms are computed entirely in spectral space. The code is tailor-made for the particular flow (3). Because ψ has only a single spectral component, the spectral algorithm in this case is comparable to pseudospectral methods in terms of computational efficiency, although, in general, pseudospectral methods are much more efficient. In our calculations, the tracer field is truncated at either 64×64 (for $\mathrm{Pe} \leq 100$) or 128×128 waves. The solution is advanced in time using a leapfrog method with a Robert filter. All calculations have also been repeated using a finite-difference code, and the two codes agree within a few percent.

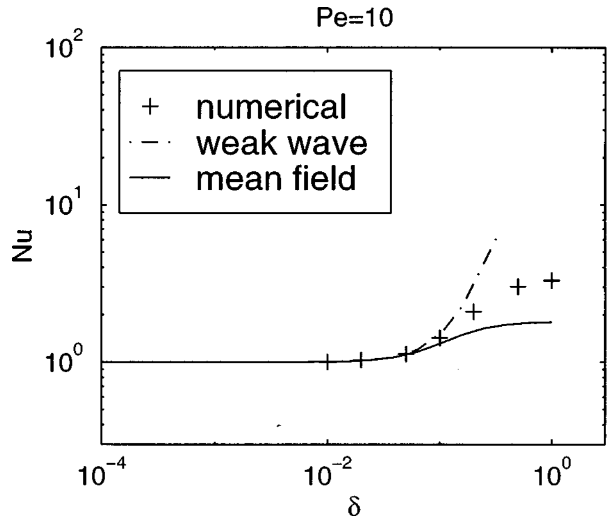

FIG. 1. Nu vs δ, for Pe = 10. Full numerical solutions are shown for larger values of δ, while the weak-wave asymptotic analytical solution is shown for the smaller values of δ. Mean-field results are shown throughout the entire range.

B. Results

1. Nu vs δ: Validity of mean-field theory

We are interested in steady state behavior. We initialize all calculations with a tracer distribution

$$q = y$$

and run until the tracer field is steady in a reference frame moving with the phase speed c. Figures 1, 2, and 3 present Nu as a function of δ, for Pe = 10,100,1000, respectively. The weak-wave asymptotic and mean-field solutions are shown together with the full numerical solutions. As might be expected, the maximum value of Nu is an increasing func-

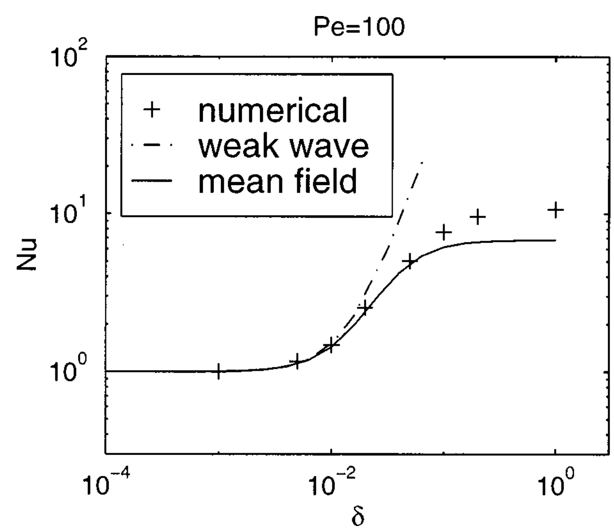

FIG. 2. As in Fig. 1, but for Pe = 100.

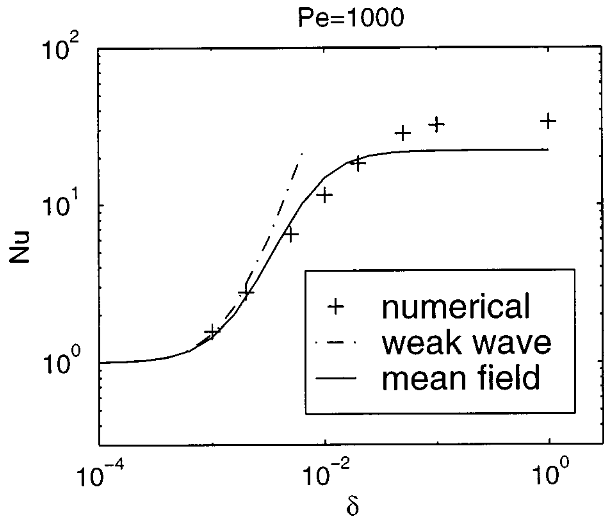

FIG. 3. As in Fig. 1, but for Pe = 1000.

tion of Pe. Also unsurprisingly, the weak-wave solution is accurate at sufficiently low δ so that $\epsilon = \mathrm{Pe}\,\delta \ll 1$.

In the numerical solutions, focusing on the Pe = 1000 case, Nu changes very little as a function of δ in the range $0.1 < \delta < \infty$ (the result for $\delta = \infty$ cannot be shown on the plot but is virtually identical to the result for $\delta = 1$), indicating that the regime $\delta \gtrsim 0.1$ is little different from the case of a nonpropagating wave. The mean-field solutions, fairly accurate for smaller δ, plateau at a somewhat smaller value of Nu, differing from the numerical solutions by roughly 48%, 36%, and 34% of the latter at Pe = 10,100,1000, respectively.

2. Nu vs ϵ: Validity of Flierl and Dewar solution

Figure 4 shows Nu vs ϵ, both as computed numerically for fixed Pe = 1000 and according to the asymptotic analytical solution of Flierl and Dewar. The agreement between the numerical and asymptotic solutions is good for $\epsilon < 10$, but the two diverge at larger ϵ.

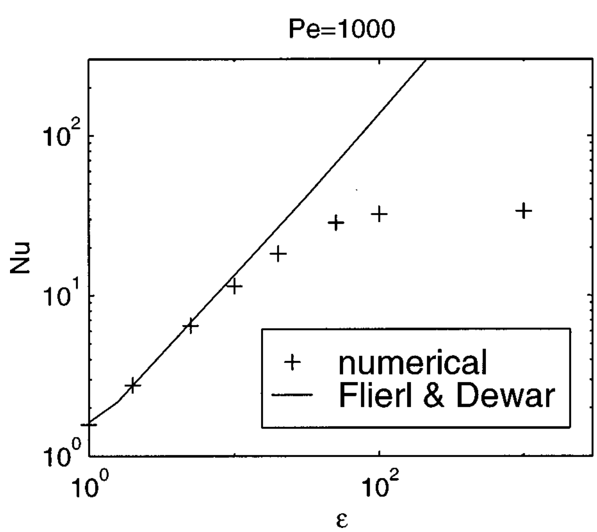

FIG. 4. Nu vs ϵ. The asymptotic analytical solution of Flierl and Dewar is shown, as well as the numerical solutions. The numerical solutions use fixed Pe = 1000.

We should expect the Flierl and Dewar solution to fail for sufficiently large finite ϵ. Even apart from any arguments involving the value of δ, there are a number of ways of seeing this, simply by considering the case $c = 0$ (implying infinite ϵ for finite fluid velocity) and assuming that it is not a singular limit. In this case $\Psi = 0$ at both $y = 0$ and $y = 1$, and therefore $q = q(\Psi)$ cannot satisfy the boundary conditions. From another point of view, it is easily shown (and the explicit calculations verify) that in the Flierl and Dewar solution for this flow, $\mathrm{Nu} \approx \epsilon$ for large ϵ—this follows more or less directly from the fact that in this limit all the tracer gradient is concentrated in boundary layers of width ϵ^{-1}—yet we know that a nonpropagating wave ($\epsilon = \infty$) has finite Nu. However, neither of these arguments tells us at what finite ϵ the failure will begin to occur. The argument presented by Flierl and Dewar, on the other hand, suggests that their solution should be valid as long as $\delta \ll 1$, or $\epsilon \ll \mathrm{Pe}$, but Fig. 4 shows that this is not the case. For example at $\epsilon = 100$, an order of magnitude less than Pe, the difference between the numerical and analytical solutions is roughly a factor of 3.

The discrepancy is resolved by the following argument concerning the consistency of the Flierl and Dewar solution with the scaling used to obtain it. Consider the order of magnitude of each term in (4). At first glance it would seem that indeed the condition $\delta \ll 1$ is sufficient to justify neglect of the diffusion term. However, in Flierl and Dewar's solution, the tracer gradient is all concentrated in boundary layers of width ϵ^{-1}. For consistency, in the region of those boundary layers we should consider the Laplacian in the diffusion term to scale as ϵ^2. Therefore, if we take Flierl and Dewar's asymptotic solution and plug it back into (4), we know that in the boundary layers the appropriately scaled equation is

$$\frac{\partial \Psi}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial \Psi}{\partial y}\frac{\partial q}{\partial x} = \epsilon^2 \delta \nabla^2 q. \tag{9}$$

Note that the spatial derivatives in the advective terms are unchanged relative to (4) in the boundary layers, because the flow is along tracer isopleths in Flierl and Dewar's asymptotic solution.

Clearly, the diffusion term in (9) cannot be neglected relative to the advective terms when $\delta \sim \epsilon^{-2}$ or greater, or equivalently when $\epsilon \sim \mathrm{Pe}^{1/3}$ or greater. Since the numerical calculations in Fig. 4 were performed at Pe = 1000, this implies failure of the Flierl and Dewar solution at $\epsilon \sim 10$, as is borne out by the calculations.

V. DISCUSSION

The primary contribution of the present work is in illustrating the transition in the cross-channel transport from the case of nonpropagating waves ($c = 0$, $\epsilon = \delta = \infty$), that is, stationary overturning cells, to that of fast traveling waves. The calculations reveal a regime which we may think of as containing slow waves, that is, waves whose periods are long compared to the characteristic advection time (large ϵ), yet which still may be short compared to the characteristic diffusion time (fairly small δ). The cross-channel flux due to these slow waves is nearly the same as that for the stationary case. The inability of the fast-wave solution of Flierl and Dewar to capture this regime is explainable, as discussed above, in terms of the self-consistency of that solution with the approximations made in deriving it.

The mean-field approximation does better at capturing the solution for slow (and stationary) waves, but still is somewhat inaccurate, predicting values of Nu which are too low by amounts on the order of 30%–50%. This can be understood in terms of the known character of the solutions for $c=0$, which is retained to some extent in the slow-wave regime. The tracer field in these cases tends to be homogenized throughout the channel interior, with gradients confined to boundary layers along the walls and narrow tongues extending across the channel along those streamlines where (in the stationary case) the x velocity vanishes. Such tracer fields clearly will not be as well represented by a single Fourier mode as will tracer fields in the fast-wave regime. The latter have a more wavelike character, as we know since Flierl and Dewar's solution tells us that in this regime the tracer isopleths and (open) streamlines must closely resemble each other.

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