Theor. Comput. Fluid Dyn. (2006)
DOI 10.1007/s00162-006-0028-8

**ORIGINAL ARTICLE**

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# Nonlinear shallow-water solutions using the weak temperature gradient approximation

Received: 6 August 2005 / Accepted: 25 April 2006


**Abstract** A form of the weak temperature gradient (WTG) approximation, in which the temperature tendency and advection terms are neglected in the temperature equation so that the equation reduces to a diagnostic balance between heating and vertical motion, is applied to a two-dimensional nonlinear shallow-water model with the heating (mass source) parameterized as a Newtonian relaxation on the temperature (layer thickness) towards a prescribed function of latitude and longitude, containing an isolated maximum or minimum, as in the classic linear Gill problem. In this model, temperature variations are retained in the Newtonian heating term, so that it is not a pure WTG system. It contains no free unbalanced modes, but reduces to the Gill model in the steady linear limit, so that steady solutions may be thought of as containing components corresponding to unbalanced modes in the same sense as the latter. The equations are solved numerically and are compared with full shallow-water solutions in which the WTG approximation is not made. Several external parameters are varied, including the strength, location, sign, and horizontal scale of the mass source, the Rayleigh friction coefficient, and the time scale for the relaxation on the mass field. Indices of the Walker and Hadley circulations are examined as functions of these external parameters. Differences between the WTG solutions and those from the full shallow-water system are small over most of the parameter regime studied, which includes time-dependent as well as steady solutions.

**Keywords** Tropical circulation · Balance model · Weak temperature gradient approximation

**PACS** 92.60.Bh

## 1 Introduction

In the dynamical meteorology of the extratropics, the notion of balanced dynamics has been extremely powerful, with the quasi-geostrophic model being the paradigmatic example [15]. The weak temperature gradient (WTG) approximation has been proposed [26,27] as a general balance model for the tropics, following earlier work developing the idea in narrower contexts [4,5,14,19]. Majda and Klein [18] define several different variants of the WTG system. All share the common feature that, in the temperature equation, the tendency and horizontal advection of temperature are neglected, consistent with the smallness of horizontal temperature gradients of the tropics. This is the observed consequence of geostrophic adjustment under small Coriolis parameter. The neglect of the temperature tendency filters out gravity waves, directly analogous to extratropical balance models such as quasi-geostrophy. The key balance in WTG is between diabatic and adiabatic heating terms in the temperature equation, whereas the key balance in extratropical balance models is between the Coriolis and pressure gradient terms in the momentum equation.

Communicated by R. Klein

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These tropical balance models do not, on their own, have the explanatory power of quasi-geostrophy, because the tropical flow is so strongly influenced by deep moist convection, whose parameterization is a problem separate from the formulation of the balance models [1]. At the same time it is useful to understand which component of the tropical circulation is balanced. In one application of particular interest, the WTG approximation can be used as a parameterization of large-scale dynamics to force simulations with single-column models (e.g., Sobel and Bretherton 2000; [7]) or limited-area cloud-resolving models [22], which are important tools in addressing the convective parameterization problem. This methodology allows these models to be used to address questions that are otherwise off limits to them, since it allows the slowly varying component of the model's precipitation field to respond to forcings such as sea surface temperature and vertical wind shear, something that previously existing formulations do not allow. Given that this methodology for the small-scale simulation implies an approximation to the equations of motion on the larger scale, it is desirable to understand the ramifications of that approximation by itself. The current work is a step towards doing that.

The WTG approximation has already been shown to be consistent with particular widely accepted idealized models for certain aspects of the tropical flow. A classical model for the zonally averaged tropical circulation is built on the framework of axisymmetric, nonlinear flow [9–11,14,20,23]. WTG solutions have been shown to agree well with numerical solutions of the full equations and to converge to the traditional angular-momentum-conserving solutions in the inviscid limit in the context of this model (Polvani and Sobel, 2002). In contrast, for the zonally asymmetric component of the flow, linear theory has been viewed as satisfactory, and has been used to explain the equatorial Walker circulation [12]. The WTG approximation to the Gill model retains the basic qualitative features of the original, and in the limit of weak thermal damping is quite a good approximation to it [3].

Here, we investigate the problem between that of the linear Gill model and axisymmetric, nonlinear Hadley cell model. We consider the nonlinear shallow-water equations forced by a mass source formulated as a Newtonian relaxation to a background equilibrium state. The equilibrium state is allowed to be a function of both spatial dimensions, so that the induced flow is not axisymmetric. Some aspects of this problem were investigated by Hsu and Plumb [16] for the case of a positive mass source, representing upper tropospheric outflow from monsoon convection. Their numerical calculations show that. with sufficiently large east–west asymmetry induced either by the $\beta$ effect or an imposed zonal flow, and sufficiently small viscosity, the forced anticyclone is distorted, becomes unstable and periodically sheds eddies, a feature they also found to be evident in observations. There are also studies of nonlinear effects on heat-induced circulation [13] that use the linear Gill model as a starting point and examine how the solutions are distorted as the amplitude of the forcing is increased. Schneider [24] started from a nonlinear axisymmetric solution, and used it as a basic state for linear calculations to determine the asymmetric aspects of the modified Hadley circulation.

Our interest here is in exploring the parameter space of this problem more fully, and in using it as a test bed to examine the fidelity of WTG solutions to solutions of the full shallow-water equations. Our approach covers the Gill model and the axisymmetric nonlinear model as two limits – choosing weak heating together with strong Rayleigh damping, and making the equilibrium temperature profile independent of zonal coordinate – and also provides a comparable framework to that of Hsu and Plumb [16]. We hope this more general framework can help us to move one step closer to a coherent balance theory for the entire tropical circulation.

The system considered here is not a pure WTG system, in the sense that all temporal and spatial variations of temperature (or here, strictly, layer thickness) are neglected in the temperature equation. Rather, in our system, the tendency and advection terms are neglected in that equation, but balanced temperature variations are retained in the parameterized Newtonian heating term, which is temperature dependent. Our system is balanced in the sense that it contains no free gravity waves, but in the steady, linear limit, it reduces to the Gill [12] model, not to the strict WTG version of the Gill model considered by Bretherton and Sobel [3]. In the sense that components of the Gill model solution are traditionally identified with unbalanced, e.g., Kelvin modes, our model may also be considered to be partly unbalanced, in that it contains these same components in its steady solutions, despite the absence of any free unbalanced modes.

## 2 Model

In the language of Majda and Klein [18], our WTG system is mesoscale, which we nonetheless use for planetary-scale simulations and show to be a good approximation to the full shallow-water equations. The inclusion of a mass source that depends on the layer thickness (the temperature-like variable) itself brings a new aspect to the system, in that we must decide whether, and if so how, to retain the temperature variation in this dependence. Under the most strict WTG scaling, it would be neglected, as in the analysis of the WTG Gill problem

[3]. In the nonlinear axisymmetric Hadley cell problem, however, retention of this dependence was found to be essential to the solution. More generally, if this dependence is not retained, the boundary conditions create a dependence on domain size, which is undesirable as described further below. Thus we retain the dependence of the mass source on the layer thickness in our WTG system. A discussion of the implied scaling and the validity of this approximation is presented in Appendix A.

We begin with the shallow-water equations on equatorial beta plane, subject to an imposed mass source and Rayleigh friction, in vorticity-divergence form:

$$\frac{\partial \zeta}{\partial t} + \nabla.(\mathbf{u}(\zeta + y)) = -\alpha\zeta \tag{1}$$

$$\frac{\partial \delta}{\partial t} + \frac{1}{2}\nabla^2(|\mathbf{u}|^2) + \frac{\partial[u(\zeta + y)]}{\partial y} - \frac{\partial[v(\zeta + y)]}{\partial x} = -\nabla^2\eta - \alpha\delta \tag{2}$$

$$\frac{\partial h}{\partial t} + \nabla.(\mathbf{u}h) = Q, \tag{3}$$

where $\mathbf{u}$ is the vector velocity, with $(u, v)$ its components in the $x$- and $y$-directions, $\zeta = \partial v/\partial x - \partial u/\partial y$ is the vorticity, $\delta = \partial u/\partial x + \partial v/\partial y$ the divergence, $h$ the layer thickness, $\eta$ the deviation of $h$ from its spatial average (see below), and $Q$ a mass source. The system here is nondimensionalized with the Rossby deformation radius $L_R = (c_{\mathrm{ref}}/\beta)^{1/2} \approx 1,500\,\mathrm{km}$ as the length scale and $T = (c_{\mathrm{ref}}\beta)^{-1/2} \approx 8\,\mathrm{h}$ as the time scale, assuming the basic state gravity-wave speed $c_{\mathrm{ref}} = 50\,\mathrm{m/s}$; $\alpha$ is the nondimensional Rayleigh friction coefficient. $\beta$ is the latitudinal gradient of the Coriolis parameter at the equator, $\beta = 2\Omega/a$, where $\Omega$ is the rotation rate of the planet and $a$ its radius.

We separate the fluid depth into a global mean without spatial variability and fluctuating perturbation,

$$h = H(t) + \eta(\mathbf{x}, t), \tag{4}$$

where $\mathbf{x}$ denotes horizontal position. The heating (mass source) is parameterized as a Newtonian relaxation on the layer thickness towards a prescribed function of latitude and longitude, containing an isolated maximum or minimum, as in the classic linear Gill problem. That is,

$$Q = \frac{h_e - (H + \eta)}{\tau}, \tag{5}$$

where $h_e$ is a specified equilibrium temperature profile, a function of both $x$ and $y$, and $\tau$ is the thermal relaxation time scale. In this study, we choose the particular form

$$h_e = H_e + \eta_e = H_e + Q_0 \exp\left[-\frac{(y - y_0)^2}{b^2} - \frac{(x - x_0)^2}{b^2}\right], \tag{6}$$

where $H_e = 1$.

To apply WTG, we neglect $\eta$ on the left-hand side of the layer thickness equation to derive a balance between horizontal divergence and the spatially varying component of the heating:

$$H\delta = Q - \bar{Q}, \tag{7}$$

where $\bar{Q}$ is the domain average of $Q$,

$$\bar{Q} = \frac{\bar{h}_e - H}{\tau},$$

where the overbar represents the domain average.

The mean mass source must be removed because the boundary conditions require that the divergence integrated over the domain vanish. We must then separately solve the spatially averaged thickness equation for the mean thickness, $H$:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \bar{Q}. \tag{8}$$

Notice that if we do not retain $\eta$ on the right-hand side of (5) – a step which is consistent in a particular scaling regime, although not exactly the one of most interest here (see Appendix A for further discussion) – the solution becomes dependent on the domain size. Steady solutions must have $H$ equal to the global mean of $h_e$ by Eqs. (5) and (8). For $h_e$ of a form like (6), with a constant term plus a deviation of nonzero mean, this requires $h \neq h_e$ at the boundaries in $y$, with the difference depending on the domain size, since for a fixed deviation [here, the second term in (6)], the global mean of $h_e$ will decrease as the domain size increases. Through (5), this means that the forcing will vary, at least slightly, with the domain size, and there will be finite divergence at the boundaries in $y$ even if those boundaries are far from the region of significant variation in $h_e$, an unphysical result which does not occur in solutions to the full shallow-water system (1)–(3). When $\eta$ is retained in the forcing, this does not happen. The total thickness is able to relax smoothly towards $H_e$ in the far field, and for sufficiently large domain sizes (in the $y$-direction in particular) the solutions become independent of that size, and the divergence becomes vanishingly small near the boundaries.

Notice also that the relaxation time scale, $\tau$, is a key parameter that we expect to control the validity of the WTG approximation. In the limit as $\tau$ vanishes, (3) and (5) implies $h = h_e$, so for sufficiently small $\tau$ we expect $h$ variations to approach those in $h_e$ in magnitude. Arguably, WTG is not a meaningful approximation in this limit, since in it, $h$ variations are not small compared to the the value they would have for no dynamics (though it may still be true that, for example, $\mathbf{u} \cdot \nabla h \ll H \nabla \cdot \mathbf{u}$). Most of our simulations are therefore carried out for relatively large $\tau$; our control value is $\tau = 8.64$, corresponding to approximately 3 days. In our model, which lacks moist physics, the Newtonian relaxation term stands in for both deep convection, with a time scale of hours, and radiation, with a time scale of weeks. Our choice lies intermediate between these two.

When the WTG approximation is made, the divergence is determined from $h$ in Eq. (7). If we use this equation to substitute for the divergence in the divergence Eq. (2), and use the explicit form for $Q$, we obtain a prognostic equation for $\eta$:

$$\frac{\partial \eta}{\partial t} - \tau H \bigtriangledown^2 \eta = \frac{1}{2} \tau H \bigtriangledown^2 (|u|^2) - \tau H \frac{\partial [v(\zeta + y)]}{\partial x} + \tau H \frac{\partial [u(\zeta + y)]}{\partial y} + \tau \alpha Q', \tag{9}$$

where $Q' = Q - \bar{Q}$, and we have neglected terms in $\mathrm{d}H/\mathrm{d}t$, as they are small compared to $\partial \eta / \partial t$. Our WTG system consists of equations (7), (8), (9) and the vorticity equation (1), which is unmodified. In both systems, $u$,$v$ are related to $\delta$, $\zeta$ through the stream function $\psi$ and the velocity potential $\chi$ by

$$\bigtriangledown^2 \chi = \delta, \tag{10}$$

$$\bigtriangledown^2 \psi = \zeta, \tag{11}$$

$$u = -\psi_y + \chi_x, \tag{12}$$

$$v = \psi_x + \chi_y. \tag{13}$$

The boundary conditions are periodic in $x$ for all fields. In $y$, we use rigid walls at $y = \pm L_y/2$, where $L_y$ is the total domain size in $y$. We apply a kinematic boundary condition at the walls,

$$v = 0. \tag{14}$$

This is enforced by requiring that the divergent and rotational parts of $v$ vanish separately there,

$$\frac{\partial \chi}{\partial y} = \frac{\partial \psi}{\partial x} = 0. \tag{15}$$

This is not the most general boundary condition, since strictly, only the sum of the divergent and rotational parts must vanish. We apply a free-slip condition on $u$,

$$\frac{\partial u}{\partial y} = 0, \tag{16}$$

which together with (14) implies that the vorticity itself must vanish at the walls,

$$\zeta = 0. \tag{17}$$

Finally, a boundary condition on $\eta$ is needed. We require that $\eta$ be in geostrophic balance at the walls,

$$\frac{\partial \eta}{\partial y} = -yu. \tag{18}$$

We have experimented with different boundary conditions (for example, $\eta = 0$ at the walls) and found no sensitivity, presumably because our forcing is sufficiently localized near the equator, and our domain sufficiently large that the flow essentially vanishes at the walls and for some distance away from them.

It would be consistent with the notion of WTG as a balance approximation to neglect the $\partial\delta/\partial t$ term in the divergence Eq. (2), resulting in an elliptic problem for $\eta$, or similarly to neglect the $\eta$ tendency in Eq. (9), as seems appropriate for $\tau H \gg 1$. We choose here to keep the time derivative, thus solving a diffusion-type equation (9) for $\eta$ prognostically in time, mainly because it is numerically easier. Neglecting $\eta$ in (9) will lead to a diagnostic equation between $\eta$ and $\mathbf{u}$,

$$\nabla^2\eta = -\frac{1}{2}\nabla^2(|u|^2) + \frac{\partial[v(\zeta + y)]}{\partial x} - \frac{\partial[u(\zeta + y)]}{\partial y} - \frac{\alpha Q'}{H}, \tag{19}$$

leaving only one prognostic equation (1) in the whole system. Calculation of $\mathbf{u}$ requires knowledge of the divergence, $\delta$, which can only be found through Eq. (7). Calculation of RHS of Eq. (7), however, requires $\eta$, through Eq. (5). Computation of $\eta$, on the other hand, requires $\mathbf{u}$ through Eq. (19). The two variables $\eta$ and $\mathbf{u}$ are strongly coupled, via multiple diagnostic equations. Solution of this system requires either an iterative method, or elimination of one of the coupled variables, resulting in a higher-order diagnostic equation, which would also require appropriate boundary conditions. We refer to Eq. (9) as a diffusion-type equation, in the sense that it has the form of a standard diffusion equation, plus additional terms, with a diffusivity coefficient of $\tau H$. Formally, retaining the time derivative on $\eta$ results in an additional linear mode, compared to a 'strict' WTG system that would have only Rossby modes (Sobel et al. 2001). We show in an appendix that the mode introduced by this time derivative is not a gravity mode, and is in any case strongly stable (strongly decaying in time) and thus does not represent a source of unbalanced disturbances.

There are five external parameters in this system: the location, the size and strength of the equilibrium temperature, the relaxation time scale, and the Rayleigh friction coefficient.

## 3 Results

### 3.1 Experiment outline

We now present nonlinear numerical simulations of both the WTG and full shallow-water equation (SWE) systems. The SWE system consists of Eqs. (1)–(3) while our WTG system consists of Eqs. (1), (7)–(9); the elliptic problem (10)–(13) is common to both. As stated above, we consider a localized Gaussian equilibrium temperature profile

$$\eta_e = Q_0 \exp\left(-\frac{(y - y_0)^2}{b^2} - \frac{(x - x_0)^2}{b^2}\right). \tag{20}$$

We use a zonally periodic domain and two rigid walls on the northern and southern boundaries where the meridional velocity goes to zero. We have confirmed by experimentation that the model domain in all experiments is sufficiently large that the boundary conditions do not affect the results. The domain has 151 × 151 grid points with a grid spacing of 0.133 $L_R$ for steady-state runs and a grid point separation of 0.04 $L_R$ for time-dependent solution runs, so the latter have a much smaller domain (as well as a forcing of much smaller horizontal scale, see below).

To solve hyperbolic problems in our system of PDEs, a semi-Lagrangian scheme is used where the advection process is expressed in Lagrangian form with second-order Adams–Bashforth treatment of source terms [8]. The subroutine `hwscrt` from the Fishpack90[1] solver is used to solve the elliptic partial differential equations (PDEs); see (10) and (11). For the parabolic(diffusion-type) equation, (9), to ensure the stability with large value of $\tau$ while at the same time keeping the computation time low, an odd–even hopscotch (OEH) finite-difference scheme is used with explicit treatment of all source terms [2].

### 3.2 Two limits

The linear Gill model and axisymmetric nonlinear model become two limits of the general framework described above. First we look at how the model behaves in those two limits and compare with previous studies. Steady-state solutions are reached in these two limits.

[1] http://www.cisl.ucar.edu/css/software/fishpack90/

In the Gill limit, strong Rayleigh damping and a weak mass source (small value of $Q_0$) render the system linear, and guarantee steady solutions if the forcing is steady. In this limit, our version of WTG and the full shallow-water system are the same, both reducing to the Gill [12] model:

$$\alpha u - yv + \eta_x = 0, \tag{21}$$

$$\alpha v + yu + \eta_y = 0, \tag{22}$$

$$u_x + v_y = \frac{\eta_e - \eta}{\tau}. \tag{23}$$

As a test of our code, we used an infinitesimal value for $Q_0$ to render the solutions linear, and performed comparisons with solutions obtained from the linear code to solve the Gill model used by Bretherton and Sobel [3] using the same parameters (the full Gill model, not the model that was called the WTG version in that paper: the stricter definition of WTG, in which $\eta$ is neglected in the heating computation; here it is used). We measure the difference by the relative square errors, the square of the difference between the two solutions divided by the square of the solution itself, averaged over the domain. For all fields, these errors were a small fraction of a percent.

Next we consider the axisymmetric, or Hadley limit, as in much prior theory of the Hadley circulation [9–11,14,17,20,22,25]. To do this, we simply make the equilibrium temperature profile independent of the zonal coordinate $x$,

$$\eta_e = Q_0 \exp\left(-\frac{(y - y_0)^2}{b^2}\right) \tag{24}$$

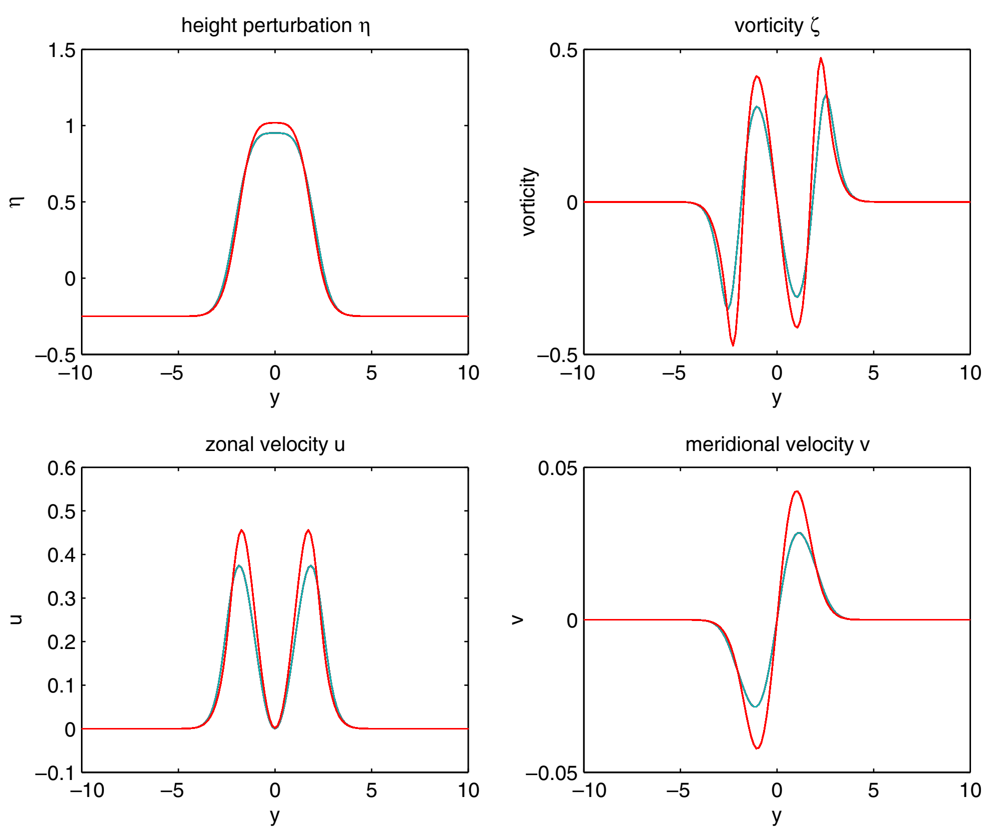


**Fig. 1** The WTG solutions (*red curves*) and the full SWE solutions (*blue curves*) for $\alpha = 0.1$, $\tau = 8.64$, $Q_0 = 2$, $b = \sqrt{2}$, $y_0 = 0$. The fields shown (from right to left and top to bottom) are perturbation layer thickness ($\eta$), vorticity ($\zeta$), zonal velocity ($u$) and meridional velocity ($v$)

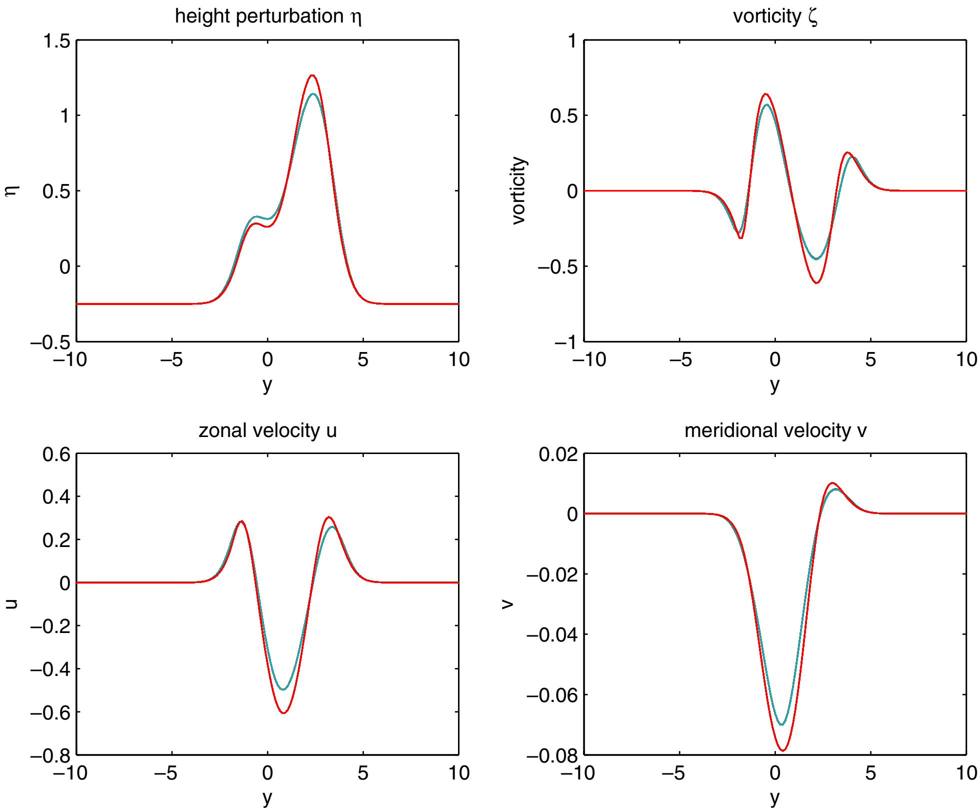


**Fig. 2** The WTG solutions (*red curves*) and the full SWE solutions (*blue curves*) for $\alpha = 0.1$, $\tau = 8.64$, $Q_0 = 2$, $b = \sqrt{2}$, $y_0 = 2$. The format is identical to that of Fig. 1

and verify after the fact that the steady solutions are in fact axisymmetric (for some parameters, it is possible to generate barotropically unstable solutions that will generate zonal asymmetries spontaneously from infinitesimal numerical noise).

We show in Fig. 1 WTG solutions for the case $\alpha = 0.1$, $\tau = 8.64, Q_0 = 2, b = \sqrt{2}$, $y_0 = 0$. The blue lines show the corresponding SWE solutions. All curves in Fig. 2 have the same parameters as in Fig. 1 except that the forcing is off-equatorial, $y_0 = 2$. In both cases, the WTG solution agrees quite well with the full shallow-water solution, especially in the sense that it is able to capture very well the size of the Hadley circulation. The WTG solutions have somewhat larger magnitude for all resulting fields. Note that our WTG solutions are not the same as those of Polvani and Sobel [21], in that they neglected spatial variability in $\eta$ in their calculation of the heating.

### 3.3 Indices of Walker and Hadley circulations

In this section, we consider solutions in the general nonlinear two-dimensional framework in which the model still reaches steady state. Relatively large values of the Rayleigh friction $\alpha$ (though still much less than unity) and spatial scale of the heating, $b$, as well as moderate values of the forcing, $Q_0$, are used. The Rossby number, a relevant measure of nonlinearity, is at most moderate, $R_0 = \frac{U}{\beta L^2} \leq 1$ (here, $\beta$ is nondimensionalized to unity, and we use $L = b$), and since there are no linear instabilities, we obtain steady solutions. To allow us to compare the models' behavior easily over a wide range of parameter space, we construct indices to represent key circulation features. The maximum value of the zonal mean meridional velocity is used to represent the Hadley circulation, while the maximum deviation of the zonal velocity on the equator from its zonal mean there is considered to represent the Walker circulation.

We examine how those two indices change with respect to the five external parameters in the system and for what parameter values and to what extent the WTG solutions differ from the SWE solutions.

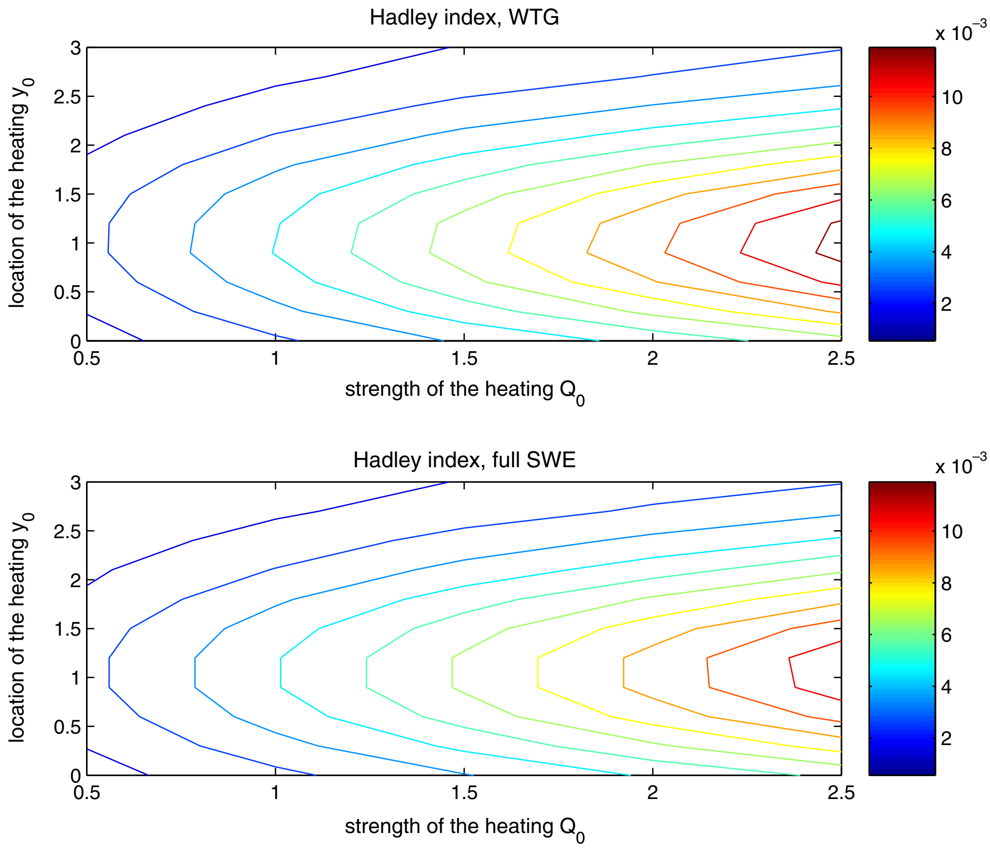


**Fig. 3** The Hadley index with respect to location and strength of the prescribed heating, for $b = 1$, $\alpha = 0.1$, and $\tau = 8.64$

### *3.3.1 Effect of variations in location and strength of heating*

In this set of experiments, parameters other than location and strength of the heating are fixed, at $b = 1$, $\alpha = 0.1$, $\tau = 8.64$

We show in Fig. 3 the Hadley index with respect to location ($y_0$) and strength ($Q_0$) of the prescribed heating. This index measures the strength of the winter cell since the maximum absolute value of the zonal mean meridional velocity occurs in the hemisphere opposite to the center of the heating. Both the full shallow-water and WTG solutions have a maximum at $y_0 \approx 1$ for all values of $Q_0$ shown. Figure 4 shows the Walker index (WI) with respect to location and strength of the prescribed heating. Both the full shallow-water solution and WTG solution show that the WI increases with the strength of heating and decreases with the location of the heating.

For both indices, the solutions are almost identical. However, note that the largest values here still correspond to only weak nonlinearity, since the relevant nondimensional measure of forcing amplitude, $Q_0/\tau$, does not exceed about 0.3 for the results shown in this figure.

Figure 5 highlights the small differences resulting from the WTG approximation by showing the dependence on $y_0$ for constant $Q_0 = 2.5$. The WTG solution overestimates both indices when the heating is placed on or close to the equator, but underestimates them when the heating is placed further away. This is true for all values of $Q_0$ we have examined.

As noted earlier, our WTG and SWE systems are identical in the steady linear limit. Thus it is desirable to increase the degree of nonlinearity, because it is there that we expect larger differences between the two systems to emerge. Figure 6 shows the Hadley and Walker indices as a function of $Q_0$ for $y_0 = 0.9$. For the largest values shown, $Q_0/\tau$ exceeds unity. As the forcing amplitude increases, the difference between the two solutions increases, as expected, due to the more significant impact of horizontal advection of $\eta$ that is neglected by WTG approximation in the continuity equation. A notable feature of both solutions is the remarkable degree of linearity in the indices as a function of $Q_0$ despite the fact that the solutions themselves have a moderate degree of nonlinearity.

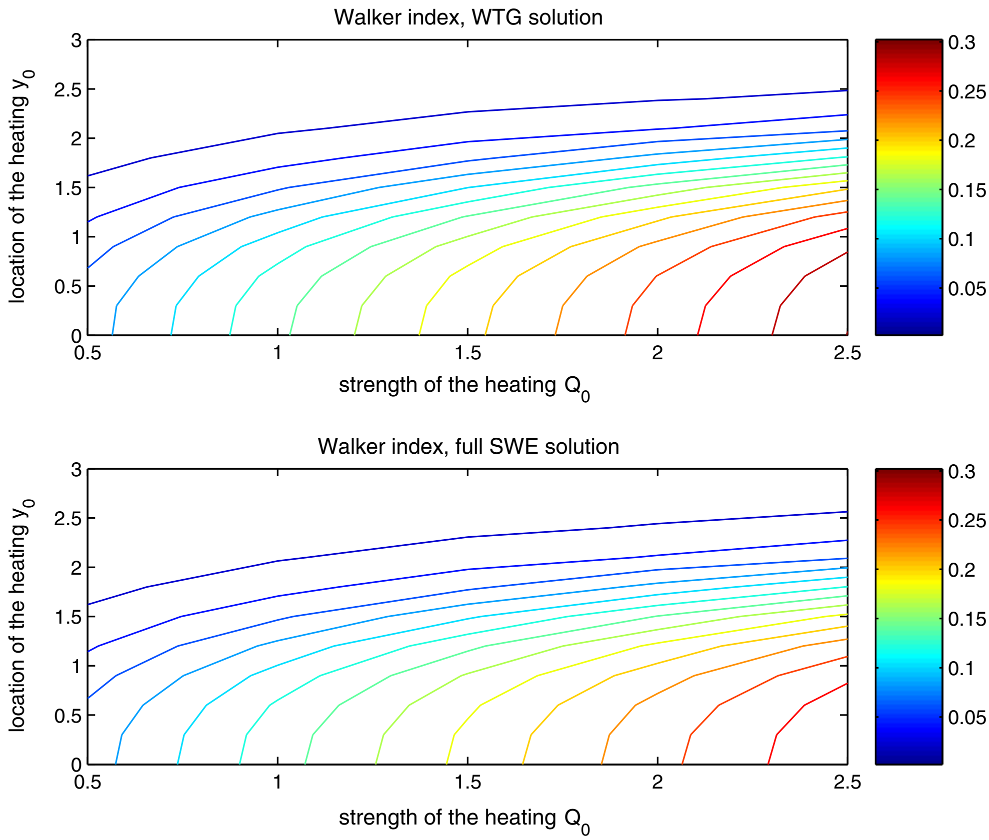


**Fig. 4** The Walker index with respect to the location and strength of the prescribed heating, for $b = 1$, $\alpha = 0.1$, and $\tau = 8.64$

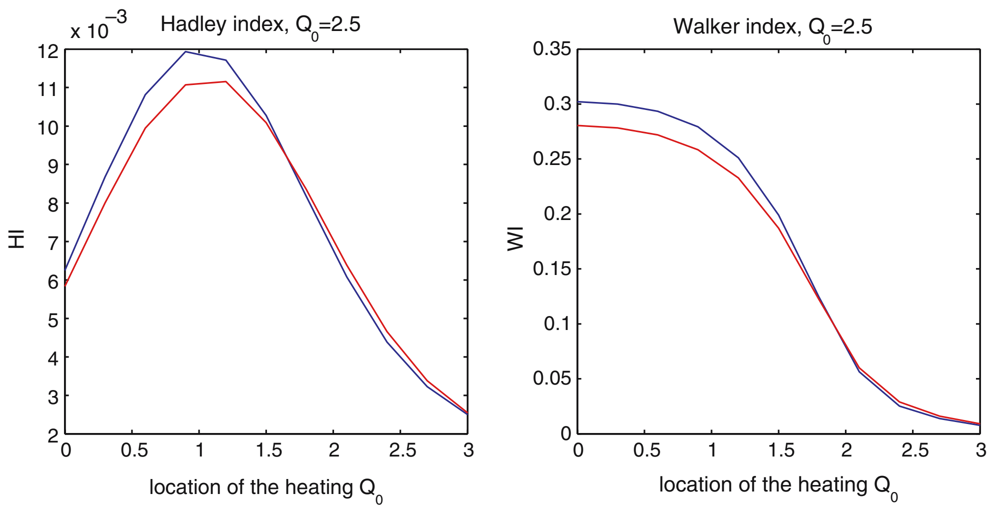


**Fig. 5** The Hadley index (*left*) and the Walker index (*right*) for the WTG solutions (*blue curves*) and the full SWE solutions (*red curves*) with respect to location of the heating, for $b = 1$, $\alpha = 0.1$, $\tau = 8.64$, and $Q_0 = 2.5$

### *3.3.2 Effect of variations in $\tau$*

Figure 7 shows the Hadley and Walker indices with respect to the Newtonian relaxation time scale, $\tau$, when the center of the heating is at 0, 0.9 and 2.0. The other parameters are $b = 1.414$, $Q_0 = 2.5$, and $\alpha = 0.1$. Both solutions show decreasing Hadley index (HI) and Walker index (WI) as $\tau$ increases. For large $\tau$, the agreement

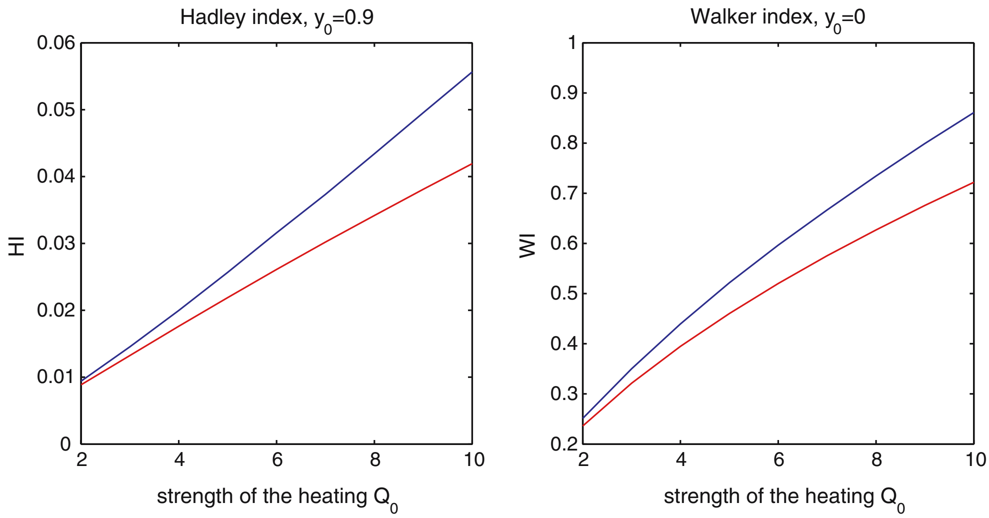


**Fig. 6** The Hadley index (*left*) and the Walker index (*right*) for the WTG solutions (*blue curves*) and the full SWE solutions (*red curves*) with respect to strength of the heating, for $b = 1$, $\alpha = 0.1$, and $\tau = 8.64$

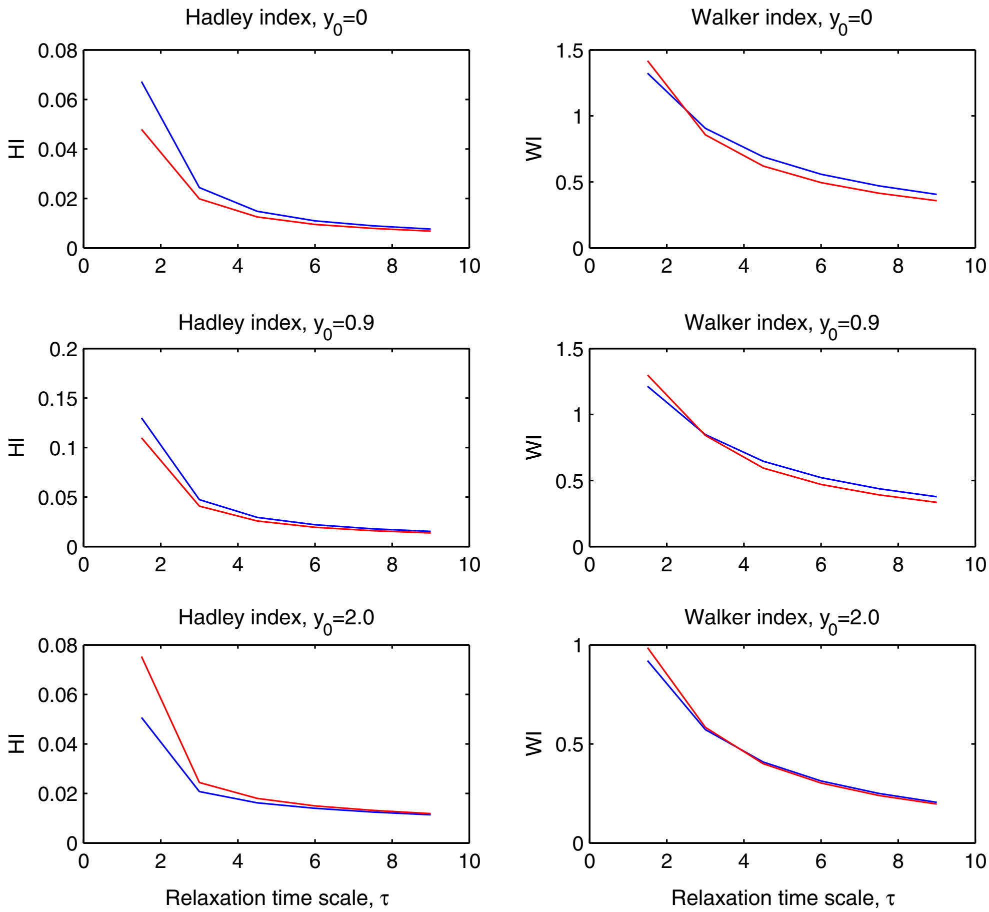


**Fig. 7** The Hadley (*left*) and the Walker (*right*) indices for the WTG solutions (*blue curves*) and the full SWE solutions (*red curves*) with respect to $\tau$ for $y_0 = 0$, 0.9, and 2.0. For all curves, $b = 1.414$, $Q_0 = 2.5$, and $\alpha = 0.1$

is quite good, while as $\tau$ becomes of order unity, the difference increases. This is to be expected, since for $\tau$ small we expect WTG to fail, as discussed earlier.

#### *3.3.3 Effect of variations in Rayleigh damping*

In this set of experiments, the Newtonian relaxation time scale is fixed to be 7.5 (2.5 days) for all plots. $Q_0$, $b$ and $y_0$ are still fixed as in the previous section. We can see that the Hadley circulation increases in strength in both solutions. The Walker circulation is weakened when $\alpha$ increases. When the center of heating is placed at $y_0 = 0$ or $y_0 = 0.9$, WTG solutions overestimate the strength of both indices for all values of Rayleigh friction (Fig. 8). When the heating is placed further away from equator at $y_0 = 2$, the WTG solutions slightly underestimate HI and WI with small Rayleigh friction, but overestimate HI and WI with large Rayleigh friction. In both cases the agreement is very close.

### 3.4 Time-dependent solution and eddy shedding behavior

Here we consider time-dependent, nonlinear solutions. The configuration here is similar to that of Hsu and Plumb [16]. Those authors discussed the parameter regime in which stable steady solutions do not exist; we access this regime here by reducing both $b$ and $\alpha$ . The parameters are $Q_0 = 1.5625$, $\tau = 8.64$, $b = 0.2$, and $y_0 = 1$, $\alpha = 0.001$, resulting in $R_o$ of $O(1)$. Figure 9 shows snapshots of absolute vorticity for both full SWE and WTG solutions. Similar features to those found in Hsu and Plumb [16] are observed. The forced anticyclone is elongated, becomes unstable and periodically sheds eddies. The two solutions are nearly identical.

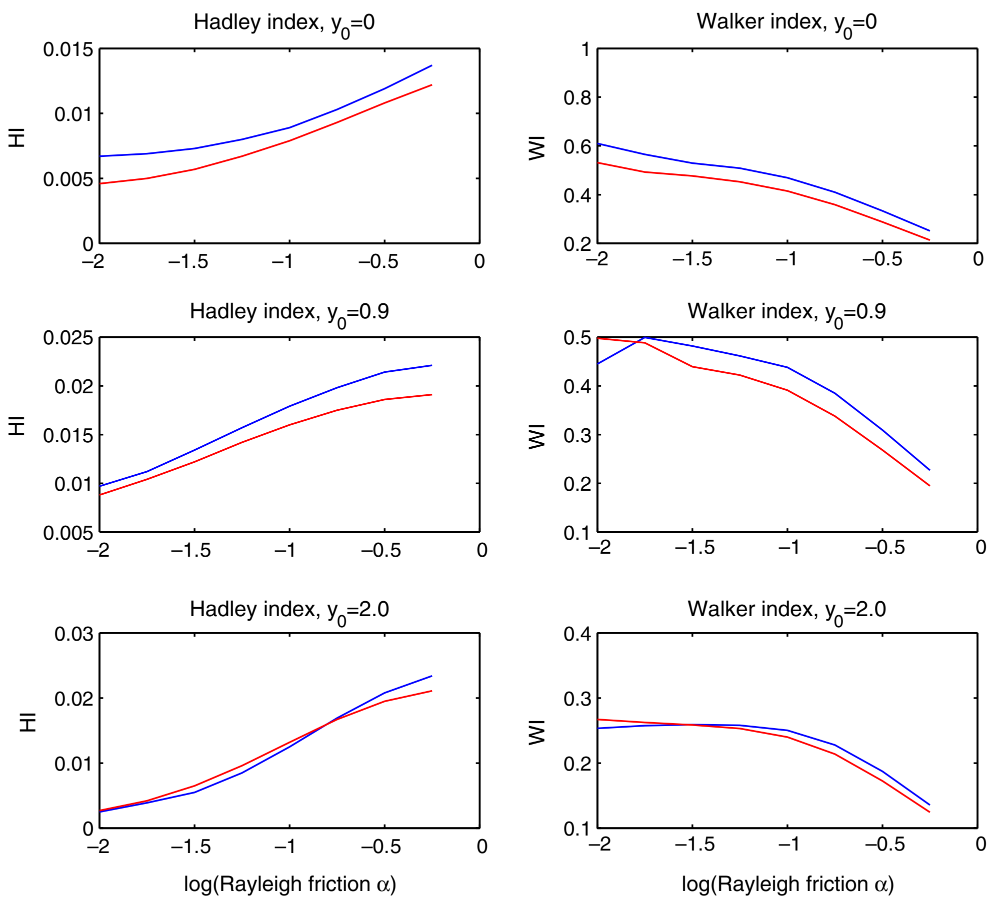


**Fig. 8** The Hadley (*left*) and the Walker (*right*) indices for the WTG solutions (*blue curves*) and the full SWE solutions (*red curves*) with respect to $\alpha$ for $y_0 = 0$, 0.9 and 2.0. For all curves, $b = 1.414$, $Q_0 = 2.5$, and $\tau = 7.5$

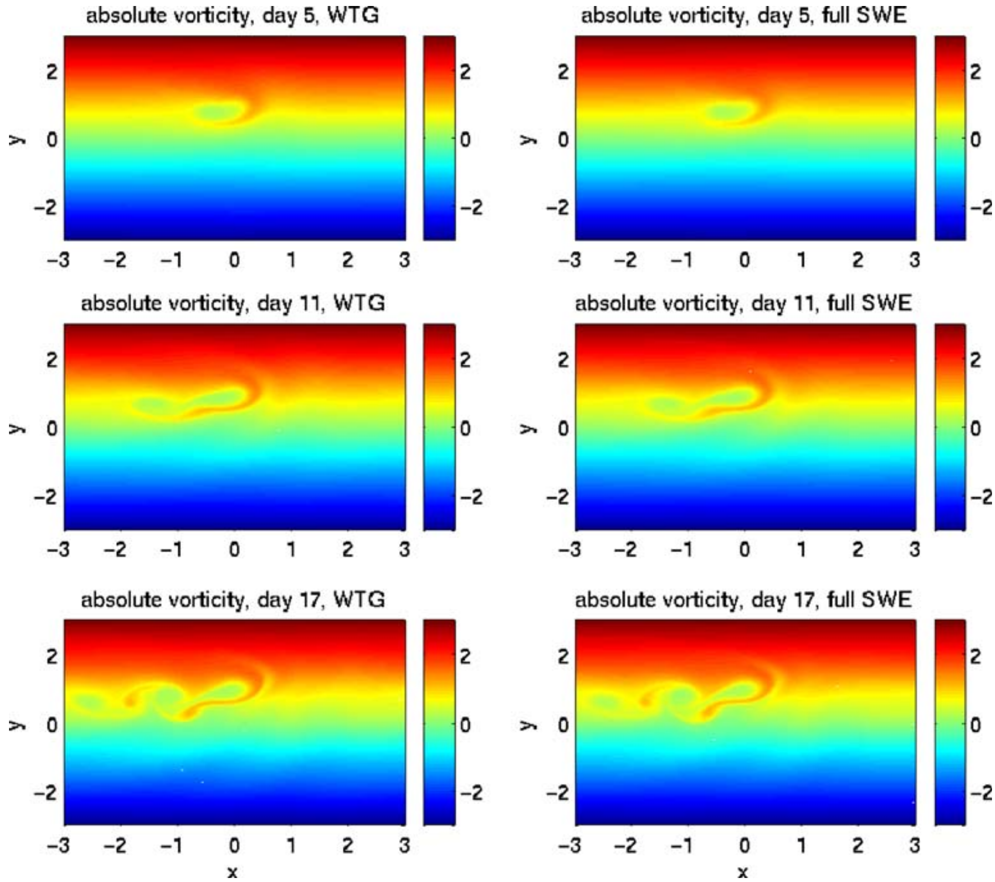


**Fig. 9** The full SWE solutions and the WTG solutions at day 5, 11, and 17 for $Q_0 = 1.5625$, $\tau = 8.64$, $b = 0.2$, $y_0 = 1$, and $\alpha = 0.001$

In this nonlinear regime, the effect of a mass sink can be expected to be different from that of a mass source. Figure 10 shows solutions for the same parameters are as in the previous case, except that now $Q_0 = -1.5625$. Again the solutions are time dependent for both the full shallow-water system and that with WTG approximation. In these solutions, a vortex spins up due to the forcing, but then propagates away due to "beta drift" [6] and eventually dissipates, while a new vortex is spun up by the forcing, a process which repeats quasi-periodically. This time-dependent behavior differs qualitatively from that in the mass source case. In that case, only secondary eddies propagate away; the primary (anticyclonic) vortex does not leave the forcing region. The cyclone in the mass sink case is also smaller and more intense than the anticyclone in the mass source case.

We can see that, in addition to preserving the basic features of weakly to moderately nonlinear steady-state SWE solutions, the WTG system is also able to capture quite well the nonlinear transient behavior of the SWE system. Close inspection does reveal small differences, however. Examination of the propagating features over long times (not shown) reveals that the propagation speed of the eddies and propagating vortex is slightly larger in the WTG than in the SWE solutions. As shown by Bretherton and Sobel [3], the dispersion relation for free Rossby waves under WTG corresponds to that from barotropic vorticity dynamics, which gives greater phase and group velocities than under the full shallow-water system with a finite deformation radius. Although the eddies here are nonlinear, Rossby wave dynamics is nonetheless involved in their propagation and we expect that this is the basic reason for the difference.

## 4 Conclusions

We have examined a form of the weak temperature gradient approximation in the context of two-dimensional equatorial nonlinear shallow-water equations with the heating parameterized as a Newtonian relaxation on

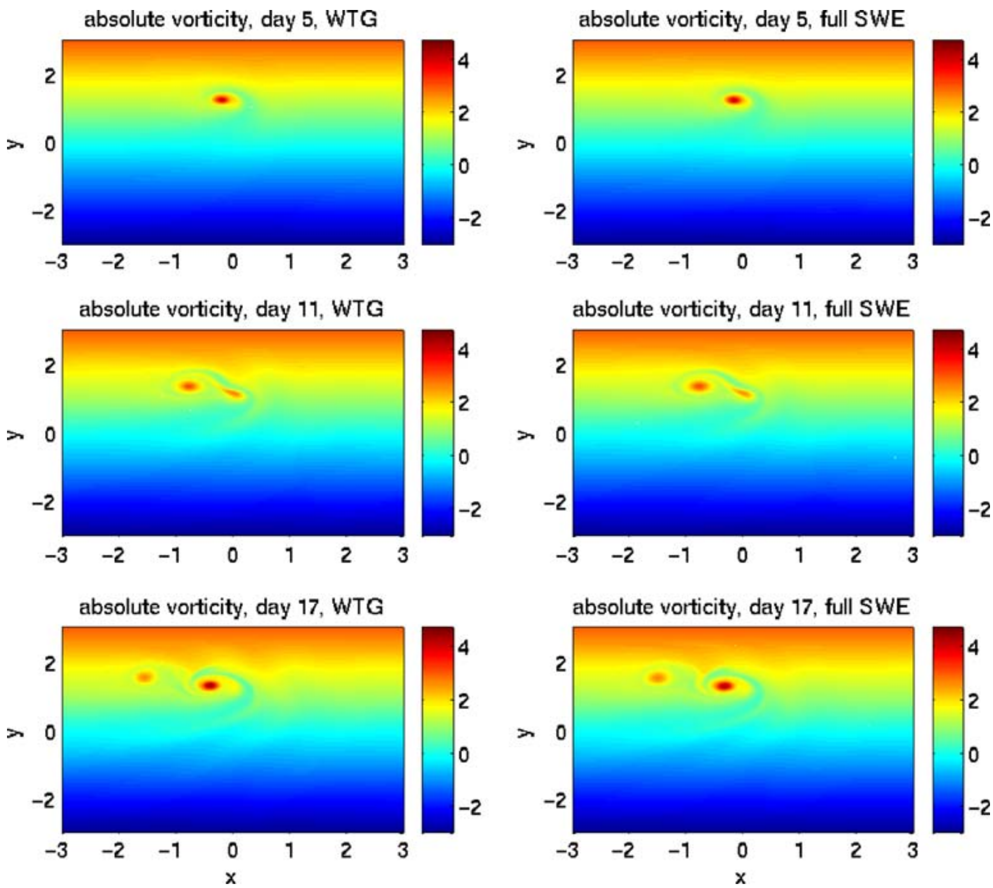


**Fig. 10** The full SWE solutions and the WTG solutions at day 5, 7, and 11 for $Q_0 = -1.5625$, $\tau = 8.64$, $b = 0.2$, $y_0 = 1$, and $\alpha = 0.001$

the layer thickness. This is an extension to previous work that has studied the WTG approximation in the context of the Gill model and axisymmetric nonlinear models of the Hadley circulation. We have found close agreement between the WTG and full shallow-water solutions over a wide range of parameter space for a more general form of the prescribed, equilibrium layer thickness field, in which the latter is a function of latitude and longitude (but not time), containing an isolated maximum or minimum. The agreement holds for both steady and transient solutions.

In the steady case in particular, for weak nonlinearity, the agreement of the two solutions is not surprising, since both the WTG and full shallow-water systems we examine reduce to the same model, that of Gill [12], in the steady, linear limit. The high degree of agreement of more nonlinear or transient solutions is perhaps less obvious, though even there, if the forcing is off the equator, we expect a predominantly rotational and thus balanced response, so the agreement is perhaps still to be expected. We have not yet performed simulations in which the equilibrium thickness itself is time dependent, but for sufficiently high frequencies this would no doubt generate strong gravity waves and cause WTG to fail.

Even where scaling arguments lead us to expect a particular approximation to be valid, there is still value in explicitly formulating the approximate system and solving it. We have done this with a particular variant of the WTG system and shown that, to the extent that this system is balanced – it has no free gravity modes, but steady solutions do have forced dissipative components of the flow that may be considered unbalanced in the sense of, e.g., the Kelvin component of the response in the Gill [12] model – a wide range of tropical flows forced by a steady equilibrium temperature field are balanced to a good approximation. Besides its conceptual value, this also provides an alternative equation set that can be used for studying the thermally forced shallow-water system on an equatorial beta plane. Because it filters gravity waves, this may be useful in some applications, and comes with little loss in accuracy, at least for the ranges of parameter space considered here.

**Acknowledgements** This work was supported by National Science Foundation grant DMS-0139830. We thank the members of the Focused Research Group on tropical dynamics for discussions.

## Appendix A: Retention of $\eta$ in the heating

To apply WTG, we neglect $\eta$ on the left-hand side of the layer thickness equation (3) to derive a balance between horizontal divergence and the spatially varying component of the heating:

$$H\delta = Q - \bar{Q}, \tag{25}$$

while we retain $\eta$ in the form of $Q$ on the right-hand side (RHS) of the equation above,

$$Q = \frac{h_e - (H + \eta)}{\tau}. \tag{26}$$

The validity of this comes down to the relative magnitudes of $\eta$ and $\eta_e$. The leading-order balance in (3) is

$$H \bigtriangledown .\mathbf{u} = Q, \tag{27}$$

which means that $\nabla \cdot \mathbf{u} \sim \frac{\eta_e}{\tau H}$; this is true whether $\eta \sim \eta_e$ or $\eta \ll \eta_e$. Neglecting $\eta$ on the LHS of (3) while retaining it in $Q$ implies that $\eta \sim \eta_e$, while both are small compared to $H$. This may be a consistent limit, and in fact we expect it to hold for $\eta_e$ and $\tau$ both sufficiently small. However, it is not the limit that is most relevant here. In the tropical atmosphere, large-scale dynamics acts to render temperature gradients small compared to what they would be in the absence of dynamics, corresponding here to the case of large $\tau$, and $\eta \ll \eta_e$. Our retention of $\eta$ in the computation of $Q$ but not elsewhere is thus based not on a fully consistent scaling argument, but on our desire to render the solutions independent of the location of the walls, as discussed in Sect. 2.

## Appendix B: Linear free mode analysis

We consider the unforced normal modes of the WTG model

$$u_t - fv + \eta_x = 0, \tag{28}$$
$$v_t + fu + \eta_y = 0, \tag{29}$$
$$\varepsilon\eta + (\partial_x u + \partial_y v) = 0 \tag{30}$$

where we imagine the damping, $\varepsilon = 1/\tau$, to be small. A single equation for $v$ may be derived:

$$(v_{xx} + v_{yy})_t - f^2\varepsilon v - \varepsilon v_{tt} + \beta v_x = 0 \tag{31}$$

We consider the mid-latitude beta plane, as this renders the analysis much simpler than on the equatorial beta plane, thus replacing $f$ in (31) by a constant, $f_0$; here we take $f_0 = 1$. Taking $v(x, y, t) = \hat{v}\exp[ikx + ily + (\sigma_r - i\sigma_i)t]$, (31) becomes:

$$-\delta_1(k^2 + l^2) - \varepsilon - \varepsilon(\sigma_r^2 - \sigma_i^2) = 0 \tag{32}$$
$$\sigma_i(k^2 + l^2) + 2\varepsilon\sigma_r\sigma_i + \beta k = 0 \tag{33}$$

First consider the nonrotating case $f_0 = \beta = 0$, solving (32) and (33) gives $\sigma_i = 0$, $\sigma_r = \frac{-(k^2+l^2)}{\varepsilon} \ll 0$. There being no rotation, there is no Rossby wave. However, there is no gravity wave either, consistent with the notion of WTG as a balance model. The only nontrivial mode in this case is a strongly damped diffusive mode with zero frequency.

For more general $f_0$ and $\beta$, eliminate $\sigma_r$ and we have

$$4\varepsilon^2\sigma_i^4 + [(k^2 + l^2)^2 - 4\varepsilon^2]\sigma_i^2 - \beta^2 k^2 = 0 \tag{34}$$

We can only keep the positive root here,

$$\sigma_i^2 = \frac{-A + \sqrt{A^2 + 16\beta^2 k^2 \varepsilon^2}}{8\varepsilon^2} = B(k, l), \tag{35}$$

where $A = (k^2 + l^2)^2 - 4\varepsilon^2$, since by assumption $\sigma_i$ is real. Hence we have two sets of solutions here corresponding to two modes with different dispersion relations:

$$\sigma_i = \pm\sqrt{B(k,l)}, \tag{36}$$

$$\sigma_r = -\frac{\beta k}{2\varepsilon\delta_2} - \frac{k^2 + l^2}{2\varepsilon}. \tag{37}$$

For $16\beta^2 k^2 \varepsilon^2 \ll A^2$, consistent with weak damping, we can further simplify the above dispersion relation using a Taylor series expansion,

$$\sqrt{A^2 + 16\beta^2 k^2 \varepsilon^2} \approx A\left(1 + \frac{8\beta^2 k^2 \varepsilon^2}{A^2}\right).$$

Then

$$\sigma_i \approx \frac{\pm\beta k}{\sqrt{(k^2 + l^2)^2 - 4\varepsilon^2}}, \tag{38}$$

$$\sigma_r = -\frac{\beta k}{2\varepsilon\delta_2} - \frac{k^2 + l^2}{2\varepsilon}. \tag{39}$$

We first consider the positive root, which has

$$\sigma_i = \frac{+\beta k}{\sqrt{(k^2 + l^2)^2 - 4\varepsilon^2}} \tag{40}$$

$$\sigma_r \ll -1. \tag{41}$$

This mode has a frequency equal in magnitude and opposite in sign to that of a Rossby wave, and so is clearly not unbalanced. That its propagation characteristics are those of a "backwards" Rossby wave might be disturbing at first glance, but this mode is very strongly damped, and so will not propagate over significant distances, to the point that it is perhaps not appropriate to think of it as a wave at all.

For the negative root,

$$\sigma_i = \frac{-\beta k}{\sqrt{(k^2 + l^2)^2 - 4\varepsilon^2}} \tag{42}$$

$$\sigma_r \approx -\varepsilon. \tag{43}$$

This is a standard weakly damped barotropic Rossby wave.

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