Renormalization group in magnetohydrodynamic turbulence

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(Received 24 September 1991; accepted 15 January 1992)

The renormalization group (RNG) theory is applied to magnetohydrodynamic (MHD) equations written in Elsässer variables, as done by Yakhot and Orszag for Navier–Stokes equations. As a result, a system of coupled nonlinear differential equations for the "effective" or turbulent "viscosities" is obtained. Without solving this system, it is possible to prove their exponential behavior at the "fixed point" and also determine the effective viscosity and resistivity. Strictly speaking, the results do not allow negative effective viscosity or resistivity, but in certain cases the effective resistivity can be continued to negative values, but not the effective viscosity. In other cases, the system tends to zero effective viscosity or resistivity. The range of possible values of the turbulent Prandtl number is also determined; the system tends to different values of this number, depending on the initial values of the viscosity and resistivity and the way the system is excited.

I. INTRODUCTION

Turbulence is one of the most challenging and least understood problems in classical physics. Fluid turbulence is usually studied by considering Navier–Stokes equations. Electrically conducting fluids, however, can contain magnetic fields and are described by magnetohydrodynamic (MHD) equations. MHD turbulence occurs in laboratory settings such as fusion confinement devices (e.g., reversed-field pinch) and astrophysical systems (e.g., solar corona). Many theories and tools used to study Navier–Stokes turbulence were adapted to MHD turbulence in view of their similarity.

The renormalization group (RNG) ideas first appeared in the 1950s in field theory.[1] Wilson's work[2] on phase transitions is the most successful application of RNG and led to numerous other papers in many different fields. Foster *et al.*[3] adapted the work of Ma and Mazenko on nonlinear spin dynamics[4] to study fluid turbulence. They considered Navier–Stokes equations driven by a random stirring force, with correlations increasing with the wave number k, and analyzed the long-term long-distance behavior of velocity correlations. More recently, Yakhot and Orszag[5,6] modified this work, reversing the wave-number dependence for the force correlations, but, in order to go further, they made controversial assumptions on the expansions about the "fixed-point" Navier–Stokes equations, which have been discussed in Refs. 7–12.

The basic idea in applying RNG to study fluid turbulence is to eliminate the smaller-scale modes, including their effect in the effective viscosity, so that only the largest scales remain. This is interesting since for high Reynolds numbers the range of scales present in Navier–Stokes turbulence is so wide that a direct numerical solution is, at present, impossible. In the case of MHD turbulence, the smaller scales are also eliminated, but their effect is incorporated in the effective viscosity and effective resistivity since there is a magnetic field present.

An application of RNG to MHD, in the manner of Foster *et al.*,[3] was reported in 1982 by Fournier *et al.*[13] Consequently, in their calculation they weighted the inertial nonlinearity and Lorentz force differently. Longcope and Sudan[14] extended the work of Yakhot and Orszag[5] to reduced MHD.

In our study we treat the full MHD equations in the manner of Yakhot and Orszag,[5] using Elsässer variables,[15] and, in contrast to Fournier *et al.*,[13] we weight all nonlinearities in the same way. Since the MHD equations contain resistivity and viscosity, both must be simultaneously renormalized and the turbulent or renormalized Prandtl number deserves special attention. In fact, its range of values can be determined by the RNG technique, which is the main result of the paper.

In the next section the MHD equations and their Fourier transform are described. Section III is devoted to the splitting into high and low wave numbers for the physical quantities and the averaging over the high wave numbers. The rescaling of the averaged equations is discussed in Sec. IV. Section V is devoted to the results of the RNG iteration and the RNG differential equations. Finally, the conclusions are presented in Sec. VI and some details of the calculations are given in the Appendices.

II. MHD EQUATIONS

The equations describing a resistive, viscous, incompressible magnetofluid are the well-known MHD equations. Stationary, isotropic MHD turbulence requires energy input to compensate for the losses due to the viscosity and resistivity. One way to do this is to add stirring random forces to the MHD equations, which, as will be seen later, allows the renormalization group (RNG) technique to be applied. The MHD equations then considered are

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + (\nabla\times\mathbf{B})\times\mathbf{B} + \nu_0\,\nabla^2\mathbf{v} + \mathbf{f}_v, \quad (1)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \eta_0\,\nabla^2\mathbf{B} + \mathbf{f}_B, \quad (2)$$

$$\nabla\cdot\mathbf{v} = 0, \quad (3)$$

$$\nabla\cdot\mathbf{B} = 0, \quad (4)$$

 0899-8221/92/051199-14$04.00

where ν_0 is the viscosity, η_0 is the resistivity and $\mathbf{f}_v$, $\mathbf{f}_B$ are the random forces. As usual, $\mathbf{v}$ is the velocity of the fluid, $\mathbf{B}$ is the magnetic field, and p is the pressure. For the sake of simplicity, the density and magnetic susceptibility are taken as units and the random forces are chosen divergence-free:

$$\nabla\cdot\mathbf{f}_v = 0,$$

$$\nabla\cdot\mathbf{f}_B = 0.$$

Using Elsässer variables,[15]

$$\mathbf{P} = \mathbf{v} + \mathbf{B}, \tag{5}$$

$$\mathbf{Q} = \mathbf{v} - \mathbf{B}, \tag{6}$$

we can rewrite the MHD equations as

$$\frac{\partial \mathbf{P}}{\partial t} + (\mathbf{Q}\cdot\nabla)\mathbf{P} = -\nabla p^* + \alpha_0 \nabla^2\mathbf{P} + \beta_0 \nabla^2\mathbf{Q} + \mathbf{f}, \tag{7}$$

$$\frac{\partial \mathbf{Q}}{\partial t} + (\mathbf{P}\cdot\nabla)\mathbf{Q} = -\nabla p^* + \alpha_0 \nabla^2\mathbf{Q} + \beta_0 \nabla^2\mathbf{P} + \mathbf{g}, \tag{8}$$

with

$$\alpha_0 = \tfrac{1}{2}(\nu_0 + \eta_0), \tag{9}$$

$$\beta_0 = \tfrac{1}{2}(\nu_0 - \eta_0), \tag{10}$$

$$\mathbf{f} = \mathbf{f}_v + \mathbf{f}_B,$$

$$\mathbf{g} = \mathbf{f}_v - \mathbf{f}_B,$$

$$p^* = p + B^2/2.$$

We introduce the Fourier decompositions of $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{f}$, $\mathbf{g}$, and p^* with an ultraviolet cutoff Λ to apply the RNG technique to Eqs. (7) and (8). In the case of $\mathbf{P}$, we have

$$\mathbf{P}(\mathbf{x},t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int_{k<\Lambda} \frac{d\mathbf{k}}{(2\pi)^d} \mathbf{P}(\mathbf{k},\omega) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)},$$

where d is the spatial dimension and our expressions are valid for $d \geqslant 2$.

As $\mathbf{P}$ and $\mathbf{Q}$, by virtue of definitions (5) and (6), have zero divergences, the Fourier transformed equations can be simplified. Indeed, if the divergence of the Fourier-decomposed MHD system is taken, p^* can be expressed in terms of $\mathbf{P}$ and $\mathbf{Q}$. With the definitions

$$\hat{k} \equiv (\omega,\mathbf{k}),$$

$$\hat{q} \equiv (\zeta,\mathbf{q}),$$

$$\int d\hat{q} \equiv \int_{-\infty}^{+\infty} \frac{d\zeta}{2\pi} \int_{q<\Lambda} \frac{d\mathbf{q}}{(2\pi)^d},$$

$$J_{lmn}(\mathbf{k}) = k_m\left(\delta_{ln} - \frac{k_l k_n}{k^2}\right) = k_m J_{ln}(\mathbf{k}), \tag{11}$$

the l component of the MHD equations is

$$G_0^{-1}(\hat{k})\begin{pmatrix} P_l(\hat{k}) \\ Q_l(\hat{k}) \end{pmatrix} = \begin{pmatrix} f_l(\hat{k}) \\ g_l(\hat{k}) \end{pmatrix} - i\lambda_0 J_{lmn}(\mathbf{k}) \begin{pmatrix} \int d\hat{q}\, Q_m(\hat{k}-\hat{q}) P_n(\hat{q}) \\ \int d\hat{q}\, P_m(\hat{k}-\hat{q}) Q_n(\hat{q}) \end{pmatrix}, \tag{12}$$

where λ_0 is the expansion parameter of the RNG technique, which at the end can be taken equal to one, and $G_0(\hat{k})$ is the Green function of system (12) as defined by its inverse,

$$G_0^{-1}(\hat{k}) = \begin{pmatrix} -i\omega + \alpha_0 k^2 & \beta_0 k^2 \\ \beta_0 k^2 & -i\omega + \alpha_0 k^2 \end{pmatrix}.$$

As in Refs. 3–5, the random forces are specified by their two-point correlations:

$$\langle f_m(\omega,\mathbf{k}) f_n(\zeta,\mathbf{q})\rangle = 2k^{-y} A_0 (2\pi)^{d+1} J_{mn}(\mathbf{k}) \delta(\omega+\zeta)\delta(\mathbf{k}+\mathbf{q}), \tag{13}$$

$$\langle f_m(\omega,\mathbf{k}) g_n(\zeta,\mathbf{q})\rangle = 2k^{-y} B_0 (2\pi)^{d+1} J_{mn}(\mathbf{k}) \delta(\omega+\zeta)\delta(\mathbf{k}+\mathbf{q}), \tag{14}$$

$$\langle g_m(\omega,\mathbf{k}) g_n(\zeta,\mathbf{q})\rangle = 2k^{-y} A_0 (2\pi)^{d+1} J_{mn}(\mathbf{k}) \delta(\omega+\zeta)\delta(\mathbf{k}+\mathbf{q}). \tag{15}$$

We consider the amplitude of the correlation $\langle fg\rangle$ to be B_0 and the amplitudes of the autocorrelations of f and g to be equal (A_0), which corresponds to $\langle f_v f_B\rangle = 0$ as in Ref. 13. Otherwise, it turns out that the RNG technique leads to a system of equations that has more terms than the original one, e.g., a $\nabla^2\mathbf{B}$ term in Eq. (1) and a $\nabla^2\mathbf{v}$ term in Eq. (2), which means that the RNG technique breaks down. This situation may be due to the fact that a finite $\langle f_v f_B\rangle$ could create coherent structures via cross helicities.

III. RENORMALIZATION GROUP APPLIED TO MHD

Our approach is the same as that applied by Yakhot and Orszag[5,6] to the Navier–Stokes equation on the basis of Foster–Nelson–Stephen theory.[3] Detailed accounts of the RNG technique can be found in Refs. 7 and 16. The functions $\mathbf{P}$ and $\mathbf{Q}$ and the forces $\mathbf{f}$ and $\mathbf{g}$ are first divided into low-wave-number and high-wave-number components:

$$P_l(\hat{k}) = \begin{cases} P_l^<(\hat{k}), & 0<k<\Lambda e^{-r}, \\ P_l^>(\hat{k}), & \Lambda e^{-r}<k<\Lambda, \end{cases} \quad r>0. \tag{16}$$

Here $Q_l^<$, $Q_l^>$, $f_l^<$, $f_l^>$, $g_l^<$, and $g_l^>$ can be defined in a similar way. The MHD system is then decomposed into

$$\begin{pmatrix} P_l^<(\hat{k}) \\ Q_l^<(\hat{k}) \end{pmatrix} = G_0^<(\hat{k}) \begin{pmatrix} f_l^<(\hat{k}) \\ g_l^<(\hat{k}) \end{pmatrix} - i\lambda_0 G_0^<(\hat{k}) J_{lmn}^<(\mathbf{k}) \begin{pmatrix} \int d\hat{q}\,[Q_m^<(\hat{k}-\hat{q}) + Q_m^>(\hat{k}-\hat{q})][P_n^<(\hat{q}) + P_n^>(\hat{q})] \\ \int d\hat{q}\,[P_m^<(\hat{k}-\hat{q}) + P_m^>(\hat{k}-\hat{q})][Q_n^<(\hat{q}) + Q_n^>(\hat{q})] \end{pmatrix}, \tag{17}$$

$$\begin{pmatrix} P_l^>(\hat{k}) \\ Q_l^>(\hat{k}) \end{pmatrix} = G_0^>(\hat{k}) \begin{pmatrix} f_l^>(\hat{k}) \\ g_l^>(\hat{k}) \end{pmatrix} - i\lambda_0 G_0^>(\hat{k}) J_{lmn}^>(\mathbf{k}) \begin{pmatrix} \int d\hat{q}[Q_m^<(\hat{k}-\hat{q}) + Q_m^>(\hat{k}-\hat{q})][P_n^<(\hat{q}) + P_n^>(\hat{q})] \\ \int d\hat{q}[P_m^<(\hat{k}-\hat{q}) + P_m^>(\hat{k}-\hat{q})][Q_n^<(\hat{q}) + Q_n^>(\hat{q})] \end{pmatrix}, \tag{18}$$

where the superscript of $G_0(\hat{k})$ and $J_{lmn}(\mathbf{k})$ means division into low- and high-wave-number components.

Our aim (with the use of compact notation) is to eliminate the $P^>$ and $Q^>$ from Eq. (17) by solving Eq. (18). The procedure can be done by expanding $P^>$ and $Q^>$ in powers of λ_0:

$$\begin{pmatrix} P_l^>(\hat{k}) \\ Q_l^>(\hat{k}) \end{pmatrix} = \begin{pmatrix} P_{l0}^>(\hat{k}) \\ Q_{l0}^>(\hat{k}) \end{pmatrix} + \lambda_0 \begin{pmatrix} P_{l1}^>(\hat{k}) \\ Q_{l1}^>(\hat{k}) \end{pmatrix} + \lambda_0^2 \begin{pmatrix} P_{l2}^>(\hat{k}) \\ Q_{l2}^>(\hat{k}) \end{pmatrix} + \cdots + \lambda_0^n \begin{pmatrix} P_{ln}^>(\hat{k}) \\ Q_{ln}^>(\hat{k}) \end{pmatrix} + \cdots. \tag{19}$$

Substituting Eq. (19) on both sides of Eq. (18) and equating the terms in powers of λ_0, we obtain up to the second order in λ_0

$$\begin{pmatrix} P_{l0}^>(\hat{k}) \\ Q_{l0}^>(\hat{k}) \end{pmatrix} = G_0^>(\hat{k}) \begin{pmatrix} f_l^>(\hat{k}) \\ g_l^>(\hat{k}) \end{pmatrix}, \tag{20}$$

$$\begin{pmatrix} P_{l1}^>(\hat{k}) \\ Q_{l1}^>(\hat{k}) \end{pmatrix} = -iJ_{lmn}^>(\mathbf{k}) G_0^>(\hat{k}) \begin{pmatrix} \int d\hat{q}[Q_m^<(\hat{k}-\hat{q}) + Q_{m0}^>(\hat{k}-\hat{q})][P_n^<(\hat{q}) + P_{n0}^>(\hat{q})] \\ \int d\hat{q}[P_m^<(\hat{k}-\hat{q}) + P_{m0}^>(\hat{k}-\hat{q})][P_n^<(\hat{q}) + Q_{n0}^>(\hat{q})] \end{pmatrix}, \tag{21}$$

$$\begin{pmatrix} P_{l2}^>(\hat{k}) \\ Q_{l2}^>(\hat{k}) \end{pmatrix} = -iJ_{lmn}^>(\mathbf{k}) G_0^>(\hat{k}) \begin{pmatrix} \int d\hat{q}[Q_m^<(\hat{k}-\hat{q}) + Q_{m0}^>(\hat{k}-\hat{q})]P_{n1}^>(\hat{q}) + Q_{m1}^>(\hat{k}-\hat{q})[P_n^<(\hat{q}) + P_{n0}^>(\hat{q})] \\ \int d\hat{q}[P_m^<(\hat{k}-\hat{q}) + P_{m0}^>(\hat{k}-\hat{q})]Q_{n1}^>(\hat{q}) + P_{m1}^>(\hat{k}-\hat{q})[Q_n^<(\hat{q}) + Q_{n0}^>(\hat{q})] \end{pmatrix}. \tag{22}$$

Substituting $P^>$ and $Q^>$ for their perturbation series (19) in Eq. (17), we have

$$\begin{aligned} \begin{pmatrix} P_l^<(\hat{k}) \\ Q_l^<(\hat{k}) \end{pmatrix} = {} & G_0^<(\hat{k}) \begin{pmatrix} f_l^<(\hat{k}) \\ g_l^<(\hat{k}) \end{pmatrix} \\ & - i\lambda_0 J_{lmn}^<(\mathbf{k}) G_0^<(\hat{k}) \begin{pmatrix} \int d\hat{q}[Q_m^<(\hat{k}-\hat{q}) + Q_{m0}^>(\hat{k}-\hat{q})][P_n^<(\hat{q}) + P_{n0}^>(\hat{q})] \\ \int d\hat{q}[P_m^<(\hat{k}-\hat{q}) + P_{m0}^>(\hat{k}-\hat{q})][Q_n^<(\hat{q}) + Q_{n0}^>(\hat{q})] \end{pmatrix} \\ & - i\lambda_0^2 J_{lmn}^<(\mathbf{k}) G_0^<(\hat{q}) \begin{pmatrix} \int d\hat{q}[Q_m^<(\hat{k}-\hat{q}) + Q_{m0}^>(\hat{k}-\hat{q})]P_{n1}^>(\hat{q}) + Q_{m1}^>(\hat{k}-\hat{q})[P_n^<(\hat{q}) + P_{n0}^>(\hat{q})] \\ \int d\hat{q}[P_m^<(\hat{k}-\hat{q}) + P_{m0}^>(\hat{k}-\hat{q})]Q_{n1}^>(\hat{q}) + P_{m1}^>(\hat{k}-\hat{q})[Q_n^<(\hat{q}) + Q_{n0}^>(\hat{q})] \end{pmatrix}, \end{aligned}$$

where $P_1^>$ and $Q_1^>$ can be expressed in terms of $P_0^>$ and $Q_0^>$ by means of Eq. (21).

The next step is to average out the effect of the high wave numbers in the shell $\Lambda e^{-r} < k < \Lambda$. The procedure is as follows.

(i) The low-wave-number components ($P^<, Q^<, f^<, g^<$) are not affected by the averaging process, i.e.,

$$\langle P_l^< \rangle = P_l^<, \quad \langle Q_l^< \rangle = Q_l^<, \quad \langle f_l^< \rangle = f_l^<, \quad \langle g_l^< \rangle = g_l^<.$$

(ii) The matrix $G_0^<$ and its elements are statistically sharp, and so the averages involving $P_0^>$ and $Q_0^>$ are calculated by means of Eq. (20) and the statistical properties of $f^>$ and $g^>$. Then, as the stirring forces have a Gaussian probability distribution, we obtain

$$\langle f_l^> \rangle = \langle g_l^> \rangle = 0,$$

$$\langle f_l^> f_m^> f_n^> \rangle = \langle f_l^> f_m^> g_n^> \rangle = \langle f_l^> g_m^> g_n^> \rangle = \langle g_l^> g_m^> g_n^> \rangle = 0,$$

and

$$\langle P_{l0}^> \rangle = \langle Q_{l0}^> \rangle = 0,$$

$$\langle P_{l0}^> P_{m0}^> P_{n0}^> \rangle = \langle P_{l0}^> P_{m0}^> Q_{n0}^> \rangle = 0,$$

$$\langle P_{l0}^> Q_{m0}^> Q_{n0}^> \rangle = \langle Q_{l0}^> Q_{m0}^> Q_{n0}^> \rangle = 0.$$

(iii) The random forces are statistically homogeneous [see Eqs. (13)–(15)], the zeroth-order high-wave-number terms depend only on the statistics of the random forces and $J_{lmn}^<(0) = 0$. Therefore all terms of the form $J_{lmn}^<(\mathbf{q})\langle P_0^>(\hat{p}) P_0^>(\hat{q}-\hat{p})\rangle$ are zero.

(iv) The third-order low-wave-number components are disregarded because they vanish as the iteration goes to the "fixed point."[3,5,7]

The result of this averaging process is

$$\begin{pmatrix} P_l^<(\hat{k}) \\ Q_l^<(\hat{k}) \end{pmatrix} = G_0^<(\hat{k}) \begin{pmatrix} f_l^<(\hat{k}) \\ g_l^<(\hat{k}) \end{pmatrix} - i\lambda_0 J_{lmn}^<(\mathbf{k}) G_0^<(\hat{k}) \times \begin{pmatrix} \int d\hat{q}\, Q_m^<(\hat{k}-\hat{q}) P_n^<(\hat{q}) \\ \int d\hat{q}\, P_m^<(\hat{k}-\hat{q}) Q_n^<(\hat{q}) \end{pmatrix} - \lambda_0^2 G_0^<(\hat{k}) \begin{pmatrix} M_1(\hat{k}) \\ M_2(\hat{k}) \end{pmatrix}. \tag{23}$$

We call the matrix

$$M(\hat{k}) = \begin{pmatrix} M_1(\hat{k}) \\ M_2(\hat{k}) \end{pmatrix}$$

the "correction matrix" and describe in Appendix A how it is calculated, the procedure is analogous to that of Navier–Stokes.[5,7] Equation (23) can be rewritten as

$$G_1^{<-1}(\hat{k}) \begin{pmatrix} P_l^<(\hat{k}) \\ Q_l^<(\hat{k}) \end{pmatrix} = \begin{pmatrix} f_l^<(\hat{k}) \\ g_l^<(\hat{k}) \end{pmatrix} - i\lambda_0 J_{lmn}^<(\mathbf{k}) \times \begin{pmatrix} \int d\hat{q}\, Q_m^<(\hat{k}-\hat{q}) P_n^<(\hat{q}) \\ \int d\hat{q}\, P_m^<(\hat{k}-\hat{q}) Q_n^<(\hat{q}) \end{pmatrix}, \tag{24}$$

where the "new" Green's function is

$$G_1^{-1}(\hat{k}) = \begin{pmatrix} -i\omega + \alpha_1 k^2 & \beta_1 k^2 \\ \beta_1 k^2 & -i\omega + \alpha_1 k^2 \end{pmatrix},$$

with

$$\alpha_1 = \alpha_0 + \frac{\lambda_0^2}{4} A_d A_0 \frac{\beta_0^2}{\alpha_0^2} \frac{1}{\nu_0^2 \eta_0^2} \frac{\Lambda^{-\epsilon}}{\epsilon} \times (e^{\epsilon r} - 1) F_1(\alpha_0, \beta_0), \tag{25}$$

$$\beta_1 = \beta_0 + \frac{\lambda_0^2}{4} A_d A_0 \frac{\beta_0^2}{\alpha_0^2} \frac{1}{\nu_0^2 \eta_0^2} \frac{\Lambda^{-\epsilon}}{\epsilon} \times (e^{\epsilon r} - 1) F_2(\alpha_0, \beta_0), \tag{26}$$

$$A_d = \frac{S_d}{(2\pi)^d} \frac{1}{d(d+2)}, \tag{27}$$

$$F_1 = (2[d^2-3] + [d-y]S) \frac{\alpha_0^4}{\beta_0^2} + ([2y+2-d] - [3d^2-8]S) \frac{\alpha_0^3}{\beta_0} + (2 - [y+d-4]S)\alpha_0^2 + (d+2)(1-[d-2]S)\alpha_0\beta_0, \tag{28}$$

$$F_2 = (2 - [y+d+4]S) \frac{\alpha_0^4}{\beta_0^2} + ([2y+d+6] - d^2 S) \frac{\alpha_0^3}{\beta_0} + (2[d^2-3] + [d-y]S)\alpha_0^2 - (d+2)(1+[d-2]S)\alpha_0\beta_0, \tag{29}$$

and $S = B_0/A_0$, $\epsilon = y - d + 4$.

It should be noted that this result was obtained based on the assumption that the effective viscosity and resistivity remain positive.

IV. RESCALING

System (24), obtained after averaging in the shell $\Lambda e^{-r} < k < \Lambda$, is very similar to the original one (12), but with k defined in the interval $0 < k < \Lambda e^{-r}$. By introducing a new variable $\tilde{k}$ such that

$$\tilde{k} = k e^r, \tag{30}$$

the system is again defined in the original interval. To compensate, the following general scalings are considered:

$$\tilde{\omega} = \omega e^{a(r)}, \tag{31}$$

$$\tilde{P}_l(\tilde{\omega}, \tilde{\mathbf{k}}) = P_l^<(\omega, \mathbf{k}) e^{-c(r)}, \tag{32}$$

$$\tilde{Q}_l(\tilde{\omega}, \tilde{\mathbf{k}}) = Q_l^<(\omega, \mathbf{k}) e^{-c(r)}, \tag{33}$$

where the functions $a(r)$ and $c(r)$ are still to be determined. In order to prevent system (24) from being modified by the rescaling, we must also have

$$\tilde{f}_l(\tilde{\omega}, \tilde{\mathbf{k}}) = f_l^<(\omega, \mathbf{k}) e^{a(r) - c(r)}, \tag{34}$$

$$\tilde{g}_l(\tilde{\omega}, \tilde{\mathbf{k}}) = g_l^<(\omega, \mathbf{k}) e^{a(r) - c(r)}, \tag{35}$$

$$\tilde{\alpha}(r) = \alpha_1 e^{a(r) - 2r},$$

$$\tilde{\beta}(r) = \beta_1 e^{a(r) - 2r},$$

$$\tilde{\lambda}(r) = \lambda_0 e^{c(r) - (d+1)r}.$$

The way the stirring forces work on the system must be unaffected by this procedure, and therefore the correlation of the rescaled stirring forces must be kept equal to the old ones. This requirement is only met if[7]

$$2c = 3a + (y+d)r.$$

Then, the rescaled system is

$$\tilde{G}_1^{-1}(\tilde{\hat{k}}) \begin{pmatrix} \tilde{P}_l(\tilde{\hat{k}}) \\ \tilde{Q}_l(\tilde{\hat{k}}) \end{pmatrix} = \begin{pmatrix} \tilde{f}_l(\tilde{\hat{k}}) \\ \tilde{g}_l(\tilde{\hat{k}}) \end{pmatrix} - i\tilde{\lambda}(r) \tilde{J}_{lmn}(\tilde{\mathbf{k}}) \times \begin{pmatrix} \int d\tilde{\hat{q}}\, \tilde{Q}_m(\tilde{\hat{k}}-\tilde{\hat{q}}) \tilde{P}_n(\tilde{\hat{q}}) \\ \int d\tilde{\hat{q}}\, \tilde{P}_m(\tilde{\hat{k}}-\tilde{\hat{q}}) \tilde{Q}_n(\tilde{\hat{q}}) \end{pmatrix},$$

which is formally identical to system (12), showing that we can apply the RNG technique.

Using Eqs. (30)–(35), it is possible to obtain the invariance relation for the spectral energy, which is the sum of the kinetic and magnetic energies. This leads to the spectrum[3,7]

$$E(k) \simeq k^{-5/3 + 2(d-y)/3}.$$

We determine y by requiring that this spectrum fit energy spectra expected for MHD turbulence, such as the Kolmogorov spectrum,[17]

$$E(k) \simeq k^{-5/3},$$

which was obtained analytically for decaying MHD turbulence,[18] i.e., $y=d$. The phenomenological spectrum of Kraichnan,[19]

$$E(k) \simeq k^{-3/2},$$

is also considered for $y = d - \frac{1}{4}$. The choice $y = d + 2$ needed to obtain the spectrum $E(k) \simeq k^{-3}$, which appears in some cases in two dimensions,[21] is not be considered.

Fournier *et al.*[13] considered two different coefficients for the correlations of the forces (y_1 and y_2), but this is not possible in our case, because the rescaling of **P** and **Q** would have to be different from each other, and, by virtue of their definitions [Eqs. (5) and (6)], this does not make sense.

V. RNG EQUATIONS AND RESULTS

In order to eliminate a finite band of modes and be able to take the infrared limit, we iterate the procedure, eliminating an infinitesimal wave-number band at each step. With the iteration being performed as in Yakhot and Orszag,[5] Eqs. (25) and (26) can be taken as recursion relations for α_1 and β_1. By using a general procedure (see, for example, Reichl[22]) these recursion relations can be turned into differential equations:

$$\frac{d\alpha}{dr} = \frac{\lambda_0^2}{4} A_d A_0 \frac{\beta^2(r)}{\alpha^2(r)} \frac{1}{\nu^2(r)\eta^2(r)} \frac{e^{\epsilon r}}{\Lambda^\epsilon} F_1[\alpha(r),\beta(r)], \tag{36}$$

$$\frac{d\beta}{dr} = \frac{\lambda_0^2}{4} A_d A_0 \frac{\beta^2(r)}{\alpha^2(r)} \frac{1}{\nu^2(r)\eta^2(r)} \frac{e^{\epsilon r}}{\Lambda^\epsilon} F_2[\alpha(r),\beta(r)]. \tag{37}$$

A particular solution of this system of a nonlinear differential equation is

$$\alpha_p = X_1 e^{\epsilon r/3}, \tag{38}$$

$$\beta_p = X_2 e^{\epsilon r/3}. \tag{39}$$

Here X_1 and X_2 are such that

$$\tau = X_1/X_2$$

is the constant that satisfies

$$\begin{aligned}(2 - [y+d+4]S)\tau^4 + ([-2d^2+d+2y+12] \\ - [d^2+d-y]S)\tau^3 + ([-2y+2d^2+d-8] \\ + [3d^2+d-y-8]S)\tau^2 + (-[d+4] + [-d^2 \\ + y + d]S)\tau - (d+2)(1-[d-2]S) = 0.\end{aligned}$$

It can be proved that as $r \to \infty$ the ratio α/β goes asymptotically to τ, so that the exponential behavior in Eqs. (38) and (39) is not only a particular solution, but also the behavior of the system at the "fixed point." The exponential behavior of the effective viscosity and resistivity at the "fixed point" is easily obtained by means of Eqs. (9) and (10). By analogy with RNG applied to Navier–Stokes,[6] the effective viscosity and resistivity are, respectively,

$$\nu \simeq k^{-\epsilon/3},$$

$$\eta \simeq k^{-\epsilon/3}.$$

From Eqs. (36) and (37), it is possible to construct another differential equation, which is easier to analyze. Defining

$$x = \frac{\alpha}{\beta} = \frac{\nu+\eta}{\nu-\eta},$$

we obtain

$$\frac{dx}{dr} = \frac{1}{\beta}\left(\frac{d\alpha}{dr} - \frac{\alpha}{\beta}\frac{d\beta}{dr}\right). \tag{40}$$

Substituting Eqs. (36) and (37) into (40), we obtain

$$\frac{dx}{dr} = \frac{\lambda_0^2}{4} A_d A_0 \frac{e^{\epsilon r}}{\Lambda^\epsilon} \frac{\beta^2(r)}{\alpha(r)} \frac{1}{\nu^2(r)\eta^2(r)} z(x), \tag{41}$$

with

$$\begin{aligned}z(x) = (-2 + \{y+d+4\}S)x^4 + (2d^2 - d - 2y - 12 + \{d^2+d-y\}S)x^3 + (2y - 2d^2 - d + 8 \\ - \{3d^2+d-y-8\}S)x^2 + (d+4-\{-d^2+d+y\}S)x + (d+2)[1+(d-2)S].\end{aligned}$$

The sign of dx/dr is determined by the polynomial $z(x)$ since we are considering positive effective viscosity and resistivity. Therefore we look for the zeros of $z(x)$ given by

$$S = \frac{2x^4 - (2d^2-d-2y-12)x^3 - (2y-2d^2-d+8)x^2 - (d+4)x - d - 2}{(y+d+4)x^4 + (d^2+d-y)x^3 - (3d^2+d-y-8)x^2 + (d^2-d-y)x + d^2 - 4}.$$

We restrict ourselves to analysis of the region $-1 \leqslant S \leqslant 1$. This restriction can be shown to be due to the assumption $\langle f_v f_B \rangle = 0$. On the other hand, when the autocorrelations of f_v and f_B have the same amplitude, it follows that $S=0$. In the case $\langle f_v f_v \rangle = 0$, we obtain $S=1$, and for $\langle f_B f_B \rangle = 0$ we have $S = -1$. Typical plots for $S(x)$ are shown in Figs. 1 and 2. In Fig. 1, we have $d=y=3$ and in Fig. 2 $d=y=2$. Since the plots for $y = d - \frac{1}{4}$ are very similar to those for $y=d$ for every d, they are not shown here. For a certain d, the plot for $y = d - \frac{1}{4}$ differs from that of

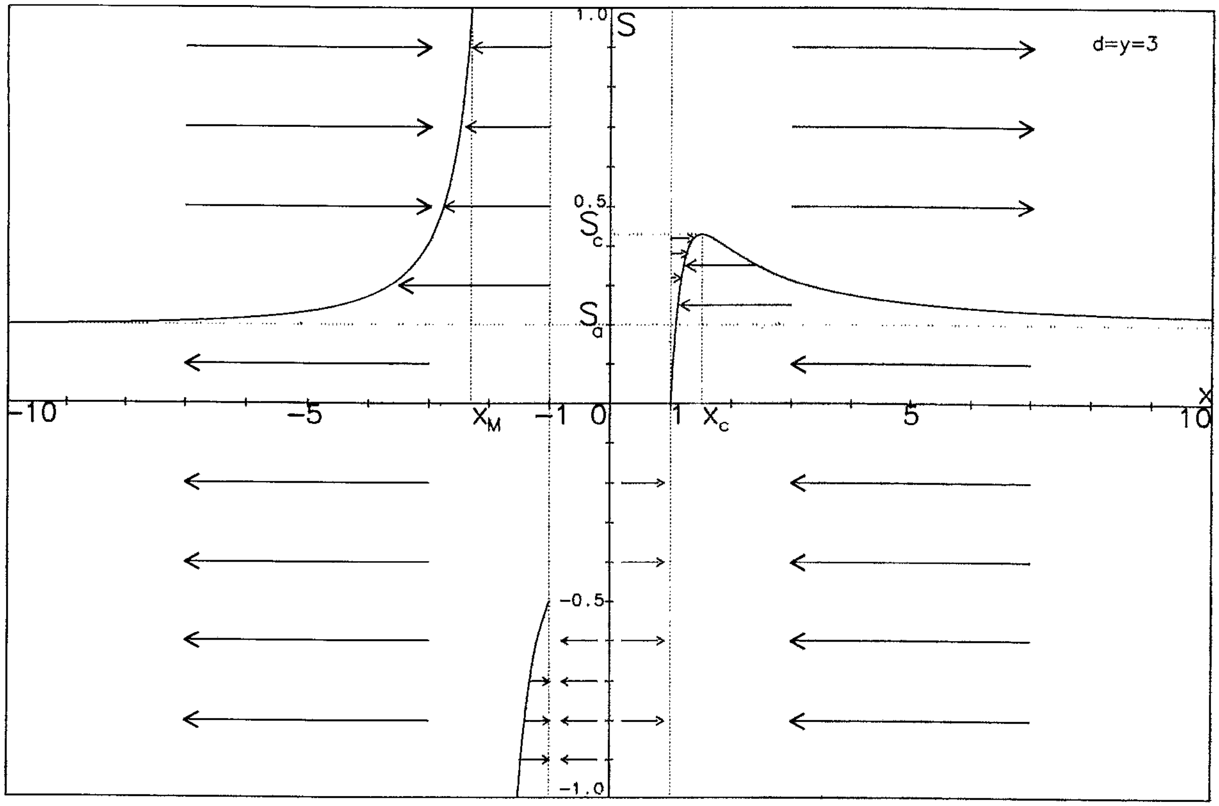

FIG. 1. Plot of $S(x)$ for $d=y=3$.

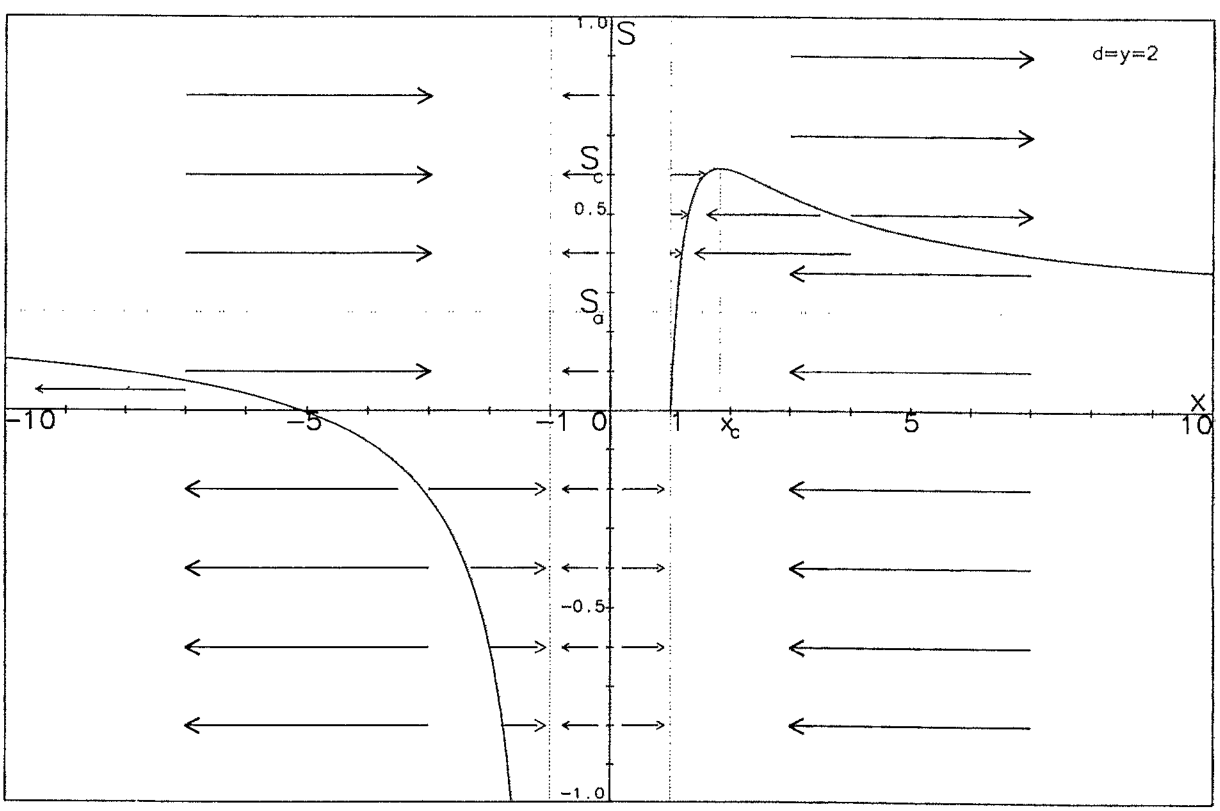

FIG. 2. Plot of $S(x)$ for $d=y=2$.

$y = d$ by just the numerical values of x in the plot, e.g., the value of x_M depends on y, but their form is exactly the same.

For a given value of S and a initial value of x_0, we obtain the direction in which the renormalized value changes as the RNG iterations proceed, using the plots $S(x)$ and the sign of $z(x)$, which gives the arrows in the plots of $S(x)$. From the arrows we are able to localize the attracting regions of the plots, which are indicated in Table I. For $d = 3$ (Fig. 1), the parameter space that tends to $x = 1$ (zero resistivity) is much larger than that leading to $x = -1$ (zero viscosity). For $x_0 > 1$, we have essentially the same behavior for $d = 3$ and $d = 2$ (Fig. 2). For $x_0 < -1$, there are differences between the behaviors of $d = 2$ and $d = 3$. For $d = 3$ (Fig. 1) there is an attracting region between the minimum of the curve at the left side (not shown in Fig. 1) and the point $(x_M, S = 1)$. This region does not appear for $d = 2$ and it is rather easy, in this case, to reach $x = -1$ since the whole curve is repelling. This does not happen for $d = 3$, because there is only a small area of the parameter space that leads to $x = -1$.

The next step is an analysis of the system, considering the possibility of negative viscosity or resistivity. In order to obtain negative values, the effective viscosity or resistivity must cross $x = 1$ (zero resistivity) or $x = -1$ (zero viscosity). When we have exactly zero resistivity (or viscosity), there is a divergence in dx/dr [see Eq. (41)] and the RNG calculation is, strictly speaking, not valid. For certain values of S and x_0, the system tends to the region of negative effective viscosity or resistivity (see Figs. 1 and 2) and we are interested in the behavior of the system once it crosses $x = 1$ or $x = -1$ and is inside this region. We then suppose that the corrected resistivity (or viscosity) is negative after a certain number of iterations (finite r) and the calculations must be redone after that owing to the change of sign, which leads to changes in the result of the integration. In Appendix B, we give more details of these calculations. The differential equation obtained for negative effective viscosity and positive effective resistivity is

$$\frac{dx}{dr} = \frac{\lambda_0^2}{16} A_0 A_d \frac{e^{\epsilon r}}{\Lambda^\epsilon} \frac{\beta(r)}{x^3(r)} \frac{1}{\nu^2(r)\eta^2(r)} R(x), \qquad (42)$$

where

$$\begin{aligned} R(x) = {} & 4[d^2 + 2d + 8 + S(d+3)]x^7 + 4[d^2 - 5d - y - 32 - S(3d+14)x^6 + 4[-3d^2 + 4d + y + 36] \\ & + S(-2d^2 + 7d + 2y + 36)]x^5 + 4[d^2 - 4d - y - 29 + S(2d^2 - 7d - 2y - 37)]x^4 \\ & + 2[8d + 2y + 50 + S(4d+9)]x^3 - (4d + 23 - 8S)x^2 + 2(2 - S)x - 19. \end{aligned}$$

In the case of negative effective resistivity and positive effective viscosity, we have

$$\frac{dx}{dr} = -\frac{\lambda_0^2}{16} A_0 A_d \frac{e^{\epsilon r}}{\Lambda^\epsilon} \frac{\beta(r)}{x^3(r)} \frac{1}{\nu^2(r)\eta^2(r)} R(x). \qquad (43)$$

The plot of $S(x)$ in the region $-1 < x < 1$ (Figs. 1 and 2) is given by equaling $R(x)$ to zero. The arrows of Figs. 1 and 2 in this region are again given by the sign of dx/dr, which is determined by $R(x)$. In the regions where the system tends to negative effective viscosity or resistivity, the function $R(x)$ is mainly negative, except in a small region for negative S and $x > 0.7$. The tendency of the system is to return to the region of positive viscosity ($|x| > 1$) owing to the sign of dx/dr near $x = -1$. This tendency is shown by the arrows in the region $-1 < x < 0$ in Figs. 1 and 2. However, once in the ($|x| > 1$) region, it tends to the negative viscosity region again, and then the system is trapped in the region of zero viscosity ($x = -1$). For $0 < x < 1$, as shown in Figs. 1 and 2, there are two possibilities for the system, for $-0.2 < S < 0$, the system is trapped in the region of zero resistivity ($x = 1$). For $-1 < S < -0.2$ there is an attractive region for $0.7 < x < 1$, which means that the system can achieve a negative resistivity. Therefore we obtained negative effective resistivity with RNG calculations, as closure theories for two-dimensional MHD[23] and reduced MHD.[24] Negative effective resistivity occurred for negative values of S (see Figs. 1 and 2), corresponding to the magnetic regime, where negative effective resistivity also appears in closure theories.[24] It must be clear that it is not possible to have initial viscosity and resistivity such that $-1 < x_0 < 1$. The arrows in this region just show the behavior of the system if it enters this region after a finite value of r, and we accept that it can cross in some way the divergence of dx/dr at $x = 1$ or $x = -1$. The fact that the system is attracted to zero effective viscosity or zero effective resistivity in some regions shows perhaps a way of annihilating the viscosity or the resistivity of the system, depending on the way the system is excited and the initial conditions. This could be interesting in some practical situations.

TABLE I. Regions of attraction for x.

$d=3$ $y=3$	$d=3$ $y=2.75$	$d=2$ $y=2$	$d=2$ $y=1.75$
$1 \leqslant x \leqslant 1.5$	$1 \leqslant x \leqslant 1.5$	$1 \leqslant x \leqslant 1.8$	$1 \leqslant x \leqslant 1.8$
$x=1$	$x=1$	$x=1$	$x=1$
$-18.4 \leqslant x \leqslant -2.3$	$-38.4 \leqslant x \leqslant -2.4$		
$x=-1$	$x=-1$	$x=-1$	$x=-1$
$0.7 \leqslant x \leqslant 1$	$0.7 \leqslant x \leqslant 1$	$0.7 \leqslant x \leqslant 1$	$0.7 \leqslant x \leqslant 1$

One physically interesting quantity being the Prandtl number

$$P_t = \nu/\eta,$$

we can relate it to x,

$$x=\frac{P_t+1}{P_t-1},$$

and obtain the possible values for the turbulent or effective Prandtl number. From Eq. (41) we obtain the corresponding differential equation for $P_t>0$,

$$\frac{dP_t}{dr}=-\frac{\lambda_0^2}{2}A_dA_0\frac{e^{\epsilon r}}{\Lambda^\epsilon}\frac{1}{(P_t+1)}\frac{1}{P_t^2\eta^3}z^*(P_t),\quad (44)$$

where

$$\begin{aligned}z^*(P_t)=2SP_t^4-[&-d^2+2d+y+12+(d^2-2d-y\\&-8)S]P_t^3-[-d^2-2d+y+4(-3d^2\\&-2d-y+4)S]P_t^2-[d^2-y-4+(d^2-y\\&-8)S]P_t-[d^2-y-4+(d^2-y-2)S].\end{aligned}$$

Using Eqs. (42) and (43) we obtain the differential equations valid for the negative effective Prandtl number, respectively, for $-1<P_t<0$ and $P_t<-1$ we have

$$\frac{dP_t}{dr}=-\frac{\lambda_0^2}{16}A_dA_0\frac{e^{\epsilon r}}{\Lambda^\epsilon}\frac{1}{(P_t+1)^3(P_t-1)}\frac{1}{P_t^2\eta^3}R^*(P_t),\quad (45)$$

$$\frac{dP_t}{dr}=\frac{\lambda_0^2}{16}A_dA_0\frac{e^{\epsilon r}}{\Lambda^\epsilon}\frac{1}{(P_t+1)^3(P_t-1)}\frac{1}{P_t^2\eta^3}R^*(P_t),\quad (46)$$

where

$$\begin{aligned}R^*(P_t)=&-3(1+4S)P_t^7+[8d^2-8d-8y-9-4(2d^2-2d-2y-7)S]P_t^6+[48d^2-16y-337\\&-8(2d^2-4d-2y-17)S]P_t^5+[104d^2+88d-8y+469+8(d^2+3d-y-2)S]P_t^4+[96d^2+160d\\&+211+4(8d^2-8y-55)S]P_t^3+[24d^2+136d+8y+889+4(2d^2+14d-2y+47)S]P_t^2+[-16d^2\\&+96d+16y+545-16(d^2-6d-y-30)S]P_t+[-8d^2+40d+8y+283-8(d^2-5d-y-23)S].\end{aligned}$$

In analogy with the procedure we did before, we make the plots of $S(P_t)$, equaling $z^*(P_t)$ and $R^*(P_t)$ to zero and analyzing the sign of dP_t/dr. These plots are given in Figs. 3–6. In Figs. 3 and 4 we consider $d=y=3$ and in Figs. 5 and 6, $d=y=2$. Figures 4 and 6 show in detail the region $-10<P_t<10$, which is not seen easily in Figs. 3 and 5. Essentially, these figures show the same result as Figs. 1 and 2, but their physical interpretation is much easier. For a certain initial Prandtl number P_{t0}, it is possible to obtain the value to which the renormalized Prandtl number tends in a way similar to that used for x. Table II shows the attractive regions of P_t. Each region of attraction of P_t shown in Table II corresponds to a region of attraction of x shown in Table I. In Figs. 3–6 we can see that $P_t=1$ is not a fixed point, although Figs. 1 and 2 suggest so. However, $P_t\approx 1$ (value considered "experimentally") is also possible in our results, depending on the initial conditions and how the system is excited. In the cases that $P_t\to\infty$, the next step is to go to the negative region through $-\infty$, this occurs as the effective resistivity must cross zero before being negative.

VI. CONCLUSION

The application of the RNG technique to MHD has brought several new features that were absent in the case of Navier–Stokes equations. First of all, the magnetic field is not a "passive vector," as noted in Ref. 13, which obliges us to renormalize simultaneously both the resistivity and viscosity. In Ref. 13, where the correlations of the stochastic stirring forces were assumed to increase toward large k, the authors had to weight the magnetic and kinetic nonlinearities in a different way. In our work the k behavior of the correlations is reversed according to Yakhot and Orszag,[5] and to the physical expectation. This leads us to weight the magnetic and kinetic nonlinearities in the same way. This circumstance makes the ordinary differential equations of RNG [see Eqs. (36) and (37)] much more involved than the Navier–Stokes case.[5]

Despite this mathematical difficulty, which prevents an explicit general solution of Eqs. (36) and (37) in closed form, as in Ref. 5, we are able to make statements about the

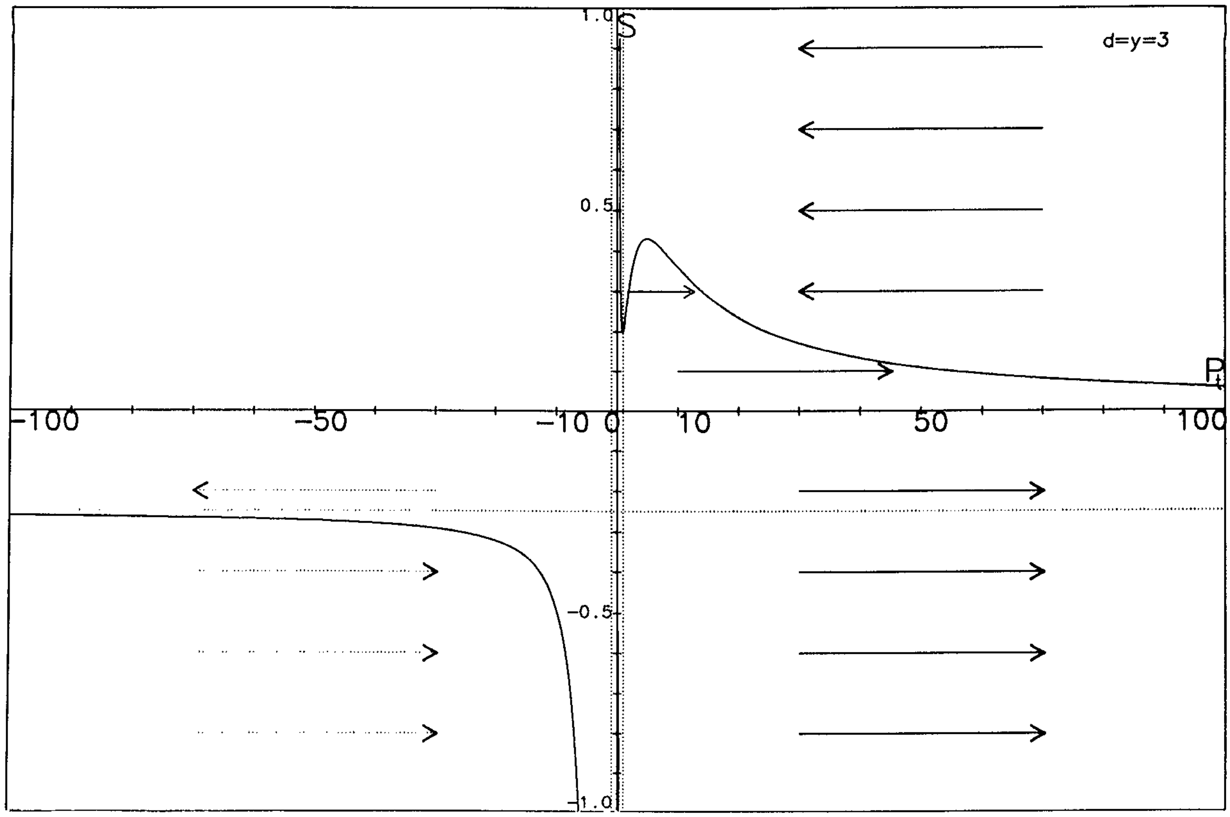

FIG. 3. Plot of $S(P_t)$ for $d=y=3$.

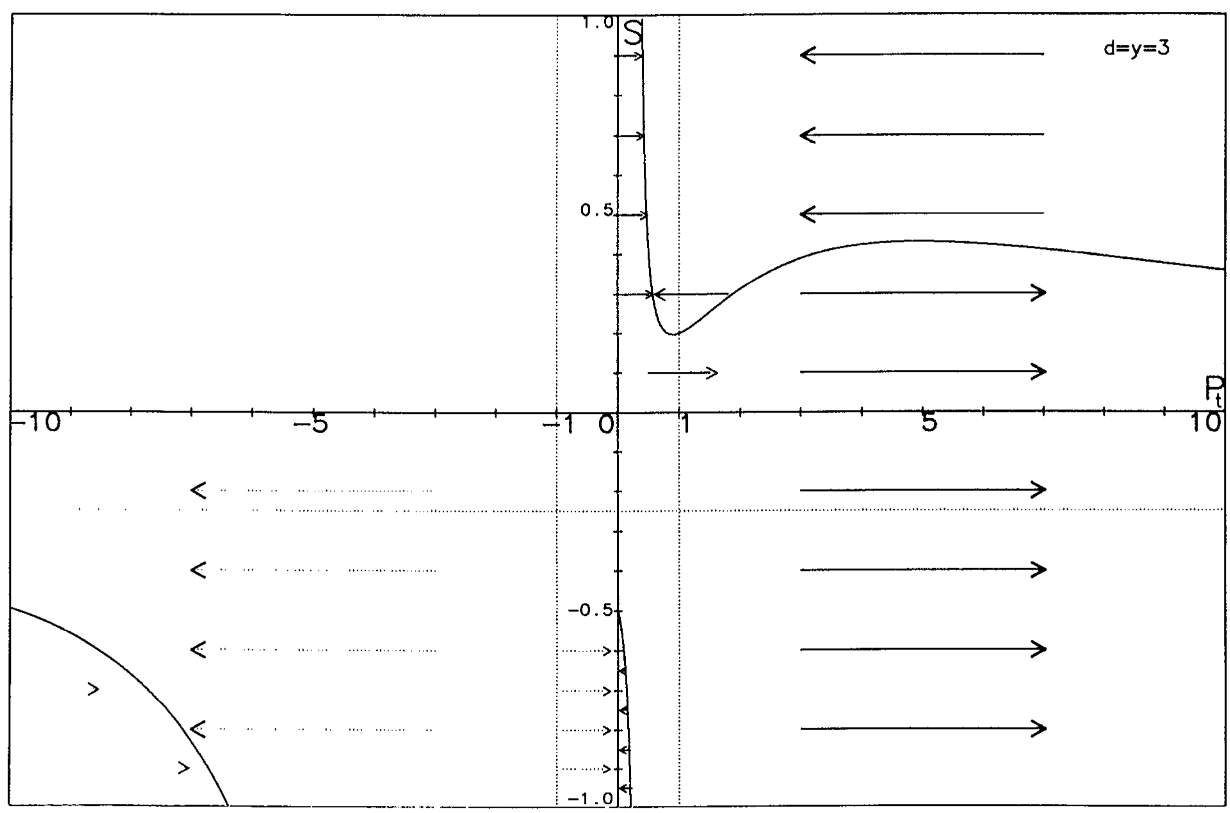

FIG. 4. Plot of $S(P_t)$ for $d=y=3$.

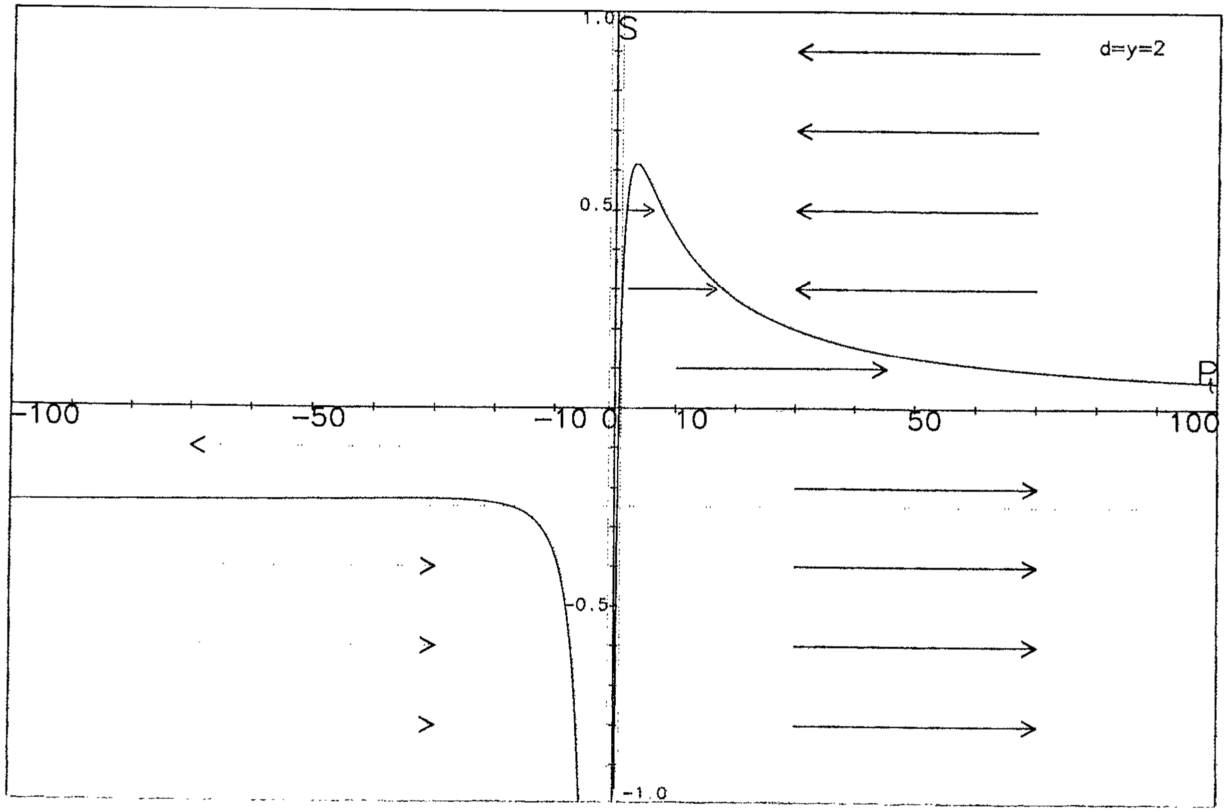

FIG. 5. Plot of $S(P_t)$ for $d=y=2$.

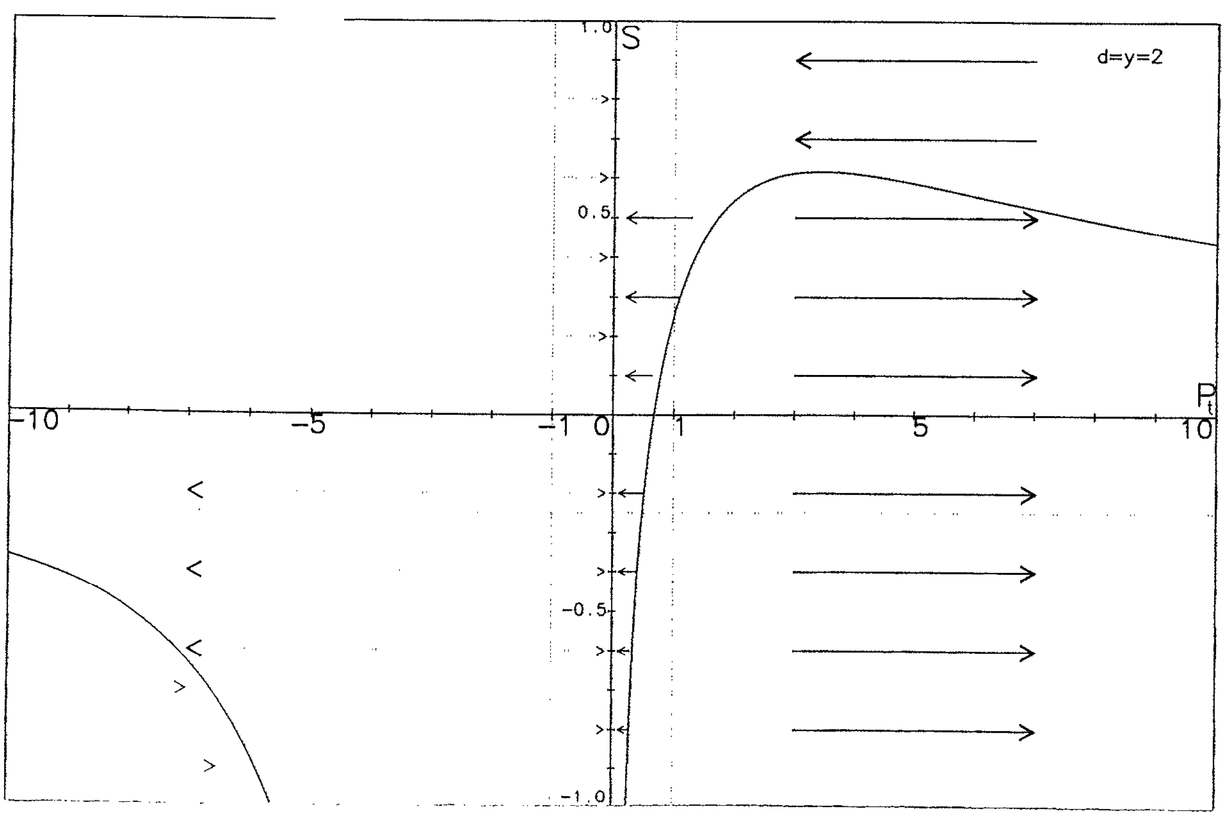

FIG. 6. Plot of $S(P_t)$ for $d=y=2$.

TABLE II. Regions of attraction for the Prandtl number P_t.

$d=3$ $y=3$	$d=3$ $y=2.75$	$d=2$ $y=2$	$d=2$ $y=1.75$
$4.9<P_t\leqslant\infty$	$4.9<P_t\leqslant\infty$	$3.4<P_t\leqslant\infty$	$3.4<P_t\leqslant\infty$
$P_t=\infty$	$P_t=\infty$	$P_t=\infty$	$P_t=\infty$
$0.4<P_t<0.9$	$0.4<P_t<0.9$		
$P_t=0$	$P_t=0$	$P_t=0$	$P_t=0$
$-\infty<P_t\leqslant-6.4$	$-\infty<P_t\leqslant-6.4$	$-\infty<P_t\leqslant-5.7$	$-\infty<P_t\leqslant-5.7$

asymptotic behavior in r of the solution and determine the effective resistivity and viscosity. In particular, it is possible to determine the attracting values of the turbulent Prandtl number (see Table II) as a function of the parameter S, which characterizes the relative correlation strength of the kinetic and magnetic stirring forces.

Note that the values of the Prandtl number do not depend upon the absolute values of the stirring forces and their correlations. Therefore statements about the turbulent Prandtl number are more likely to be representative of real turbulence, which is usually maintained by boundary conditions and not by volumetric stirring forces. This aspect is obviously absent in Navier–Stokes turbulence.

From the figures it is possible for any given S and initial values of $x_0=(\nu_0+\eta_0)/(\nu_0-\eta_0)$, with $|x_0|>1$ (or $Pt_0>0$) to see in which direction the renormalized value is going to change with the iterations of the RNG.

Negative effective viscosity is not possible in our results; instead, the tendency is to have zero effective viscosity. In certain cases, with an extended interpretation of the calculations we obtain negative effective resistivity and in others zero effective resistivity.

APPENDIX A: CALCULATION OF THE CORRECTION MATRIX

Since the calculations are very lengthy, the details will be given elsewhere. We limit ourselves here to considering two typical terms of M_1 and M_2 as follows:

$$\begin{aligned}T_1 &= J^{<}_{lmn}(\mathbf{k})\int d\hat{q}\int d\hat{p}\, J^{>}_{nrs}(\mathbf{q})z_0^{>}(\hat{q})\,[Q^{<}_{r}(\hat{q}-\hat{p})\\&\quad\times\langle Q^{>}_{m0}(\hat{k}-\hat{q})P^{>}_{s0}(\hat{p})\rangle\\&\quad+P^{<}_{s}(\hat{p})\langle Q^{>}_{m0}(\hat{k}-\hat{q})Q^{>}_{r0}(\hat{q}-\hat{p})\rangle],\end{aligned}\tag{A1}$$

$$\begin{aligned}T_2 &= -J^{<}_{lmn}(\mathbf{k})\int d\hat{q}\int d\hat{p}\, J^{>}_{mrs}(\mathbf{k}-\mathbf{q})\\&\quad\times u_0^{>}(\hat{k}-\hat{q})\,[Q^{<}_{r}(\hat{k}-\hat{q}-\hat{p})\\&\quad\times\langle P^{>}_{n0}(\hat{q})P^{>}_{s0}(\hat{p})\rangle+P^{<}_{s}(\hat{p})\\&\quad\times\langle P^{>}_{n0}(\hat{q})Q^{>}_{r0}(\hat{k}-\hat{q}-\hat{p})\rangle].\end{aligned}\tag{A2}$$

The integration over $\hat{p}$ can be performed by using the two-point correlations of the forces [Eqs. (13)–(15)]. After this integration, T_1 and T_2 are

$$\begin{aligned}T_1 &= 2J^{<}_{lmn}(\mathbf{k})\int d\hat{q}\, J^{>}_{nrs}(\mathbf{q})z_0^{>}(\hat{q})|\mathbf{k}-\mathbf{q}|^{-y}(Q^{<}_{r}(\hat{k})J^{>}_{ms}(\mathbf{k}-\mathbf{q})\{-A_0[z_0^{>}(\hat{q}-\hat{k})u_0^{>}(\hat{k}-\hat{q})\\&\quad+u_0^{>}(\hat{q}-\hat{k})z_0^{>}(\hat{k}-\hat{q})]+B_0[|z_0^{>}(\hat{k}-\hat{q})|^2+|u_0^{>}(\hat{k}-\hat{q})|^2]\}+J^{>}_{mr}(\mathbf{k}-\mathbf{q})P^{<}_{s}(\hat{k})\\&\quad\times\{A_0[|z_0^{>}(\hat{k}-\hat{q})|^2+|u_0^{>}(\hat{k}-\hat{q})|^2]-B_0[z_0^{>}(\hat{q}-\hat{k})u_0^{>}(\hat{k}-\hat{q})+u_0^{>}(\hat{q}-\hat{k})z_0^{>}(\hat{k}-\hat{q})]\}),\end{aligned}\tag{A3}$$

$$\begin{aligned}T_2 &= -2J^{<}_{lmn}(\mathbf{k})\int d\hat{q}\, J^{>}_{mrs}(\mathbf{k}-\mathbf{q})u_0^{>}(\hat{k}-\hat{q})q^{-y}(Q^{<}_{r}(\hat{k})J^{>}_{ns}(\mathbf{q})\{A_0[|z_0^{>}(\hat{q})|^2+|u_0^{>}(\hat{q})|^2]\\&\quad-B_0[z_0^{>}(\hat{q})u_0^{>}(-\hat{q})+u_0^{>}(\hat{q})z_0^{>}(-\hat{q})]\}+P^{<}_{s}(\hat{k})J^{>}_{nr}(\mathbf{q})\{-A_0[z_0^{>}(\hat{q})u_0^{>}(-\hat{q})+u_0^{>}(\hat{q})z_0^{>}(-\hat{q})]\\&\quad+B_0[|z_0^{>}(\hat{q})|^2+|u_0^{>}(\hat{q})|^2]\}).\end{aligned}\tag{A4}$$

The next step is to perform the $\hat{q}$ integration. It should be noted that, owing to the definition of the functions of high-wave numbers (16) in the integrands, the integration over $\mathbf{q}$ must be performed at the intersection of the intervals $\Lambda e^{-r}<q<\Lambda$ and $\Lambda e^{-r}<|\mathbf{k}-\mathbf{q}|<\Lambda$. This is expressed with a greater-than sign over the integral of the $\mathbf{q}$ integration,

$$\int_{-\infty}^{+\infty}\frac{d\zeta}{2\pi}\int^{>}\frac{d\mathbf{q}}{(2\pi)^d}.$$

First the integration over ζ is calculated by the residue method. In order to have a tractable contour in the complex plane, we have to assume definite signs for both the effective viscosity and resistivity. First we take positive effective resistivity and viscosity since their initial values are always positive. Calculations that consider the possibility of negative renormalized resistivity or viscosity are explained in Appendix B. We have a total of 16 different integrals in ζ to perform, e.g.,

$$\begin{aligned}I_a &= \int_{-\infty}^{\infty} d\zeta\, z_0(\hat{q})|u_0(\hat{k}-\hat{q})|^2\\&=\frac{\pi}{4}\frac{\beta_0}{\alpha_0}\frac{1}{|\mathbf{k}-\mathbf{q}|^2}\\&\quad\times\left(-\frac{-i\omega+\alpha_0q^2+\nu_0|\mathbf{k}-\mathbf{q}|^2}{[(\alpha_0q^2+\nu_0|\mathbf{k}-\mathbf{q}|^2-i\omega)^2-\beta_0^2q^4]\nu_0}\right.\\&\quad\left.+\frac{-i\omega+\alpha_0q^2+\eta_0|\mathbf{k}-\mathbf{q}|^2}{[(\alpha_0q^2+\eta_0|\mathbf{k}-\mathbf{q}|^2-i\omega)^2-\beta_0^2q^4]\eta_0}\right),\end{aligned}$$

$$I_b = \int_{-\infty}^{+\infty} d\zeta\, u_0(\hat{k}-\hat{q}) z_0(\hat{q}) u_0(-\hat{q})$$

$$= \frac{\pi}{4}\frac{\beta_0}{\alpha_0}\frac{|\mathbf{k}-\mathbf{q}|^2}{q^2}$$

$$\times\left(\frac{2\alpha_0+\beta_0}{[(\alpha_0|\mathbf{k}-\mathbf{q}|^2+\nu_0 q^2-i\omega)^2-\beta_0^2|\mathbf{k}-\mathbf{q}|^4]\nu_0} + \frac{2\alpha_0-\beta_0}{[(\alpha_0|\mathbf{k}-\mathbf{q}|^2+\eta_0 q^2-i\omega)^2-\beta_0^2|\mathbf{k}-\mathbf{q}|^4]\eta_0}\right),$$

with I_a and I_b parts of T_1 and T_2, respectively.

We are ultimately interested in terms of order k^2, which contribute to the renormalized "viscosities." Since $J_{lmn}(\mathbf{k})$ is of order k, we only need to consider the terms of the integrand up to the first order in k. In the infrared limit of interest $\omega \to 0$, an expansion in $\mathbf{k}$ is performed and we obtain for the integrals above

$$I_a = \frac{\pi}{8}\frac{\beta_0^2}{\alpha_0^2}\frac{q^{-4}}{\nu_0^2\eta_0^2}\left(3\alpha_0^2-\beta_0^2+2(4\alpha_0^2-\beta_0^2)\frac{\mathbf{k}\cdot\mathbf{q}}{q^2}\right),$$

$$I_b = \frac{\pi}{8}\frac{\beta_0^2}{\alpha_0^2}\frac{q^{-4}}{\nu_0^2\eta_0^2}\left(\alpha_0^2+\beta_0^2+2\beta_0^2\frac{\mathbf{k}\cdot\mathbf{q}}{q^2}\right).$$

These calculations lead to typical terms for M_1 and M_2, given by

$$T_{11} = \frac{1}{8}\frac{\beta_0^2}{\alpha_0^2}\frac{1}{\nu_0^2\eta_0^2} J_{lmn}^{<}(\mathbf{k}) P_s^{<}(\hat{k}) \times \int^{>} \frac{d\mathbf{q}}{(2\pi)^d} |\mathbf{k}-\mathbf{q}|^{-y} q^{-4} J_{nrs}(\mathbf{q}) \times J_{mr}(\mathbf{k}-\mathbf{q})\left(D_1+E_1\frac{\mathbf{k}\cdot\mathbf{q}}{q^2}\right),$$

$$T_{21} = \frac{1}{8}\frac{\beta_0^2}{\alpha_0^2}\frac{1}{\nu_0^2\eta_0^2} J_{lmn}^{<}(\mathbf{k}) Q_r^{<}(\hat{k}) \times \int^{>} \frac{d\mathbf{q}}{(2\pi)^d} q^{-y-4} J_{mrs}(\mathbf{k}-\mathbf{q}) \times J_{ns}(\mathbf{q})\left(D_2+E_2\frac{\mathbf{k}\cdot\mathbf{q}}{q^2}\right),$$

where T_{11} and T_{21} are parts of T_1 and T_2, respectively, and D_i and E_i are functions of α_0 and β_0.

With the definitions of J_{lmn} and J_{rs} (11), the following products of the integrands are calculated up to the first order in k:

$$J_{nrs}(\mathbf{q}) J_{mr}(\mathbf{k}-\mathbf{q}) = \left(k_m - k_r\frac{q_m q_r}{q^2}\right)\left(\delta_{ns}-\frac{q_n q_s}{q^2}\right),$$

$$J_{nrs}(\mathbf{q}) J_{ms}(\mathbf{k}-\mathbf{q}) = q_r\left(\delta_{mn}-\frac{q_m q_n}{q^2}\right) + k_n\frac{q_m q_r}{q^2} - k_s\frac{q_m q_n q_r q_s}{q^4},$$

$$J_{mrs}(\mathbf{k}-\mathbf{q}) J_{nr}(\mathbf{q}) = \left(k_n - k_r\frac{q_n q_r}{q^2}\right)\left(\delta_{ms}-\frac{q_m q_s}{q^2}\right),$$

$$J_{mrs}(\mathbf{k}-\mathbf{q}) J_{ns}(\mathbf{q}) = (k_r - q_r)\left(\delta_{mn}-\frac{q_m q_n}{q^2}\right) - \left(k_n - k_s\frac{q_n q_s}{q^2}\right)\frac{q_r q_m}{q^2}.$$

Simplifying further the "correction matrix" and noting that $J_{lmn}\delta_{mn}=0$, we can calculate the whole expression by using four different types of integrals over $\mathbf{q}$:

$$I_1 = k_m \delta_{ns}\int^{>}\frac{d\mathbf{q}}{(2\pi)^d} q^{-y-4}, \tag{A5}$$

$$I_2 = k_n \int^{>}\frac{d\mathbf{q}}{(2\pi)^d}\frac{q_r q_m}{q^2} q^{-y-4}, \tag{A6}$$

$$I_3 = \int^{>}\frac{d\mathbf{q}}{(2\pi)^d}\frac{q_m q_n q_r}{q^2} q^{-y-4}, \tag{A7}$$

$$I_4 = k_s \int^{>}\frac{d\mathbf{q}}{(2\pi)^d}\frac{q_m q_n q_r q_s}{q^4} q^{-y-4}. \tag{A8}$$

As mentioned before, the integration must be performed at the intersection of the intervals $\Lambda e^{-r} < q < \Lambda$ and $\Lambda e^{-r} < |\mathbf{k}-\mathbf{q}| < \Lambda$. Up to the first order in k, the last inequality can be written as $\Lambda e^{-r} + k\cos\gamma < q < \Lambda + k\cos\gamma$, where γ is the angle between $\mathbf{k}$ and $\mathbf{q}$. The intersection of these intervals is then

$$\Lambda e^{-r} < q < \Lambda + k\cos\gamma, \quad \cos\gamma < 0,$$
$$\Lambda e^{-r} + k\cos\gamma < q < \Lambda, \quad \cos\gamma > 0.$$

Then we have for Eq. (A5)

$$\int^{>}\frac{d\mathbf{q}}{(2\pi)^d} q^{-y-4} = \frac{k_m\delta_{ns}}{(2\pi)^d}\int d\Omega_d \int_{\Lambda e^{-r}}^{\Lambda} dq\, q^{-y-4} - \frac{k_m\delta_{ns}}{(2\pi)^d}\int d\Omega_d \int_{\Sigma} dq\, q^{-y-4} - \frac{k_m\delta_{ns}}{(2\pi)^d}\int d\Omega_d \int_{\Psi} dq\, q^{-y-4}, \tag{A9}$$

and similar expressions for Eqs. (A6)–(A8), where the domain $\Sigma = \{\Lambda e^{-r} < q < \Lambda e^{-r} + k\cos\gamma\}$ is valid for $\cos\gamma > 0$ and the domain $\Psi = \{\Lambda + k\cos\gamma < q < \Lambda\}$ is valid for $\cos\gamma < 0$. The first domain of integration in Eq. (A9) makes no contribution to the k power of the integrand, but Σ and Ψ do. The integrals I_1, I_2, and I_4 are already of first order in k, and so when calculating these integrals we only need to consider the interval $\Lambda e^{-r} < q < \Lambda$ since the other two domains make contributions to the second order in k. However, I_3 is of order zero in k and it must be calculated in the three domains, but in the first domain this integral turns out to be zero, so that only Σ and Ψ contribute to the calculation. The result for the integrals[5] is

$$I_1 = k_m\delta_{ns}\frac{S_d}{(2\pi)^d}\frac{\Lambda^{-\epsilon}}{\epsilon}(e^{\epsilon r}-1),$$

$$I_2 = k_n\delta_{rm}\frac{1}{d}\frac{S_d}{(2\pi)^d}\frac{\Lambda^{-\epsilon}}{\epsilon}(e^{\epsilon r}-1),$$

$$I_3 = -\frac{1}{2d(d+2)}\frac{S_d}{(2\pi)^d}\Lambda^{-\epsilon}(e^{\epsilon r}-1)$$
$$\times(k_r\delta_{mn}+k_n\delta_{mr}+k_m\delta_{nr}),$$
$$I_4 = \frac{k_s}{d(d+2)}\frac{S_d}{(2\pi)^d}\frac{\Lambda^{-\epsilon}}{\epsilon}(e^{\epsilon r}-1)$$
$$\times(\delta_{mn}\delta_{rs}+\delta_{mr}\delta_{ns}+\delta_{ms}\delta_{nr}).$$

The result for I_3 is exact for $d=2$, while for $d=3$ it can be proved only for certain values of γ. We did not manage to prove that it is valid for any angle γ, but due to isotropy it can be expected to be true. In the expressions above, we have $\epsilon = y+4-d$ and S_d is the area of the sphere in d dimensions, $S_d = 2\pi^{d/2}/\Gamma(d)$.[5,7]

Applying the result of the integrals in our expression and keeping in mind that $J_{lmn}\delta_{mn}=0$, we obtain an expression that can be further simplified by using

$$J^{<}_{lmn}(\mathbf{k})k_n P^{<}_m(\hat{k}) = 0,$$
$$J^{<}_{lmn}(\mathbf{k})k_n Q^{<}_m(\hat{k}) = 0,$$
$$J^{<}_{lmn}(\mathbf{k})k_m P^{<}_n(\hat{k}) = k^2 P^{<}_l(\hat{k}),$$
$$J^{<}_{lmn}(\mathbf{k})k_m Q^{<}_n(\hat{k}) = k^2 Q^{<}_l(\hat{k}).$$

The overall result for the "correction matrix" is

$$\begin{pmatrix} M_1(\hat{k}) \\ M_2(\hat{k}) \end{pmatrix} = \frac{A_0 A_d}{4}\frac{\beta_0^2}{\alpha_0^2}\frac{1}{\nu_0^2\eta_0^2}\frac{\Lambda^{-\epsilon}}{\epsilon}$$
$$\times(e^{\epsilon r}-1)k^2\begin{pmatrix} F_1 & F_2 \\ F_2 & F_1 \end{pmatrix}\begin{pmatrix} P^{<}_l(\hat{k}) \\ Q^{<}_l(\hat{k}) \end{pmatrix},$$

with

$$F_1 = -D_1 + \left(-\frac{\epsilon}{2}+y+1\right)D_2 + (d^2-3)D_3$$
$$+\left(-\frac{\epsilon}{2}+1\right)D_4 + E_4 + E_5,$$

$$F_2 = -(d^2-3)D_1 + \left(\frac{\epsilon}{2}-1\right)D_2$$
$$-E_2 + D_3 + \left(\frac{\epsilon}{2}-y-1\right)D_4 - E_6.$$

Using the expressions for D_i and E_i, we obtain Eqs. (28) and (29).

APPENDIX B: CORRECTION MATRIX FOR NEGATIVE EFFECTIVE VISCOSITY OR RESISTIVITY

Our task is to obtain an expression for the correction matrix valid for negative viscosity and positive resistivity and another valid for negative resistivity and positive viscosity.

Let us take the same typical terms of the matrix $M(\hat{k})$ calculated in Appendix A and then analyze the changes they have for negative viscosity and positive resistivity. The case of negative resistivity and positive viscosity is considered afterward.

If we choose certain initial conditions and S, then after a certain number of iterations we have zero viscosity. At this point an infinite discontinuity occurs in Eq. (41). The system would enter the region of negative viscosity, and we want to know what will happen to the effective viscosity and resistivity once we are inside this region. Therefore in this region the "initial" values of our iteration are not the molecular (initial) values, but a "corrected" positive value for the resistivity η_c and a very small negative value for the viscosity ν_c and all the old initial functions are now written as functions of these "corrected" values.

The expressions for T_1 and T_2 are the same up to the $\hat{p}$ integration [(A1)–(A4)], with just the initial values being substituted for the "corrected" values, e.g., $z_0^{<}(\hat{q})$ is now $z_c^{<}(\hat{q})$. However, the calculations change when the $\hat{q}$ integration is performed, owing to the different positions of the poles in the residue integration.

For the integrals I_a and I_b, for negative viscosity, for instance, we obtain after expanding in $\mathbf{k}$, keeping terms up to the first order and considering the infrared limit $\omega\to 0$,

$$I_a = \int_{-\infty}^{\infty} d\zeta\, z_c(\hat{q})|u_c(\hat{k}-\hat{q})|^2 = \frac{\pi}{16}\frac{\beta_c}{\alpha_c^2}\frac{q^{-4}}{\nu_c^2\eta_c^2}\left(4\alpha_c^3 - \frac{(2\alpha_c^2-2\alpha_c\beta_c+\beta_c^2)(\alpha_c-\beta_c)^2}{\beta_c}\frac{k^2}{\mathbf{k}\cdot\mathbf{q}}\right.$$
$$\left.+\frac{(2\alpha_c^6+8\alpha_c^5\beta_c-36\alpha_c^4\beta_c^2+26\alpha_c^3\beta_c^3-19\alpha_c^2\beta_c^4+8\alpha_c\beta_c^5-\beta_c^6)}{\alpha_c\beta_c^2}\frac{\mathbf{k}\cdot\mathbf{q}}{q^2}\right),$$

$$I_b = \int_{-\infty}^{+\infty} d\zeta\, u_c(\hat{k}-\hat{q})z_c(\hat{q})u_c(-\hat{q})$$
$$= \frac{\pi}{16}\frac{\beta_c}{\alpha_c^2}\frac{q^{-4}}{\nu_c^2\eta_c^2}\left(8\alpha_c^3 - 4\frac{\alpha_c^5}{\beta_c} + \frac{(\alpha_c+\beta_c)^2(2\alpha_c-\beta_c)(2\alpha_c^2-2\alpha_c\beta_c+\beta_c^2)}{\beta_c^2}\frac{k^2}{\mathbf{k}\cdot\mathbf{q}}\right.$$
$$\left.+\frac{(10\alpha_c^6+4\alpha_c^5\beta_c-8\alpha_c^4\beta_c^2+8\alpha_c^3\beta_c^3+7\alpha_c^2\beta_c^4-6\alpha_c\beta_c^5+\beta_c^6)}{\alpha_c\beta_c^2}\frac{\mathbf{k}\cdot\mathbf{q}}{q^2}\right).$$

The typical terms T_{11} and T_{21}, parts of T_1 and T_2, respectively, are then

$$T_{11}=\frac{1}{16}\frac{\alpha_c^2}{\beta_c^2}\frac{1}{\nu_c^2\eta_c^2}J^<_{lmn}(\mathbf{k})P^<_s(\hat{k})\times\int^>\frac{d\mathbf{q}}{(2\pi)^d}|\mathbf{k}-\mathbf{q}|^{-y}q^{-4}J_{nrs}(\mathbf{q})\times J_{mr}(\mathbf{k}-\mathbf{q})\left(D'_1+E'_1\frac{\mathbf{k}\cdot\mathbf{q}}{q^2}+R_1\frac{k^2}{\mathbf{k}\cdot\mathbf{q}}\right),$$

$$T_{21}=\frac{1}{16}\frac{\alpha_c^2}{\beta_c^2}\frac{1}{\nu_c^2\eta_c^2}J^<_{lmn}(\mathbf{k})Q^<_r(\hat{k})\times\int^>\frac{d\mathbf{q}}{(2\pi)^d}q^{-y-4}J_{mrs}(\mathbf{k}-\mathbf{q})\times J_{ns}(\mathbf{q})\left(D'_2+E'_2\frac{\mathbf{k}\cdot\mathbf{q}}{q^2}+R_2\frac{k^2}{\mathbf{k}\cdot\mathbf{q}}\right),$$

with D'_i, E'_i, and R_i being functions of α_c,β_c.

The integration follows the process described in Appendix A, the only difference being that now we have one more type of integral to calculate, in addition to the integrals I_1 to I_4 [(A5)–(A8)]:

$$I_5=\frac{k^2}{k_u}\int^>\frac{d\mathbf{q}}{(2\pi)^d}\frac{q_rq_mq_n}{q_uq^2}q^{-y-4}=\frac{k^2}{k_u}\frac{1}{d}\frac{S_d}{(2\pi)^d}\frac{\Lambda^{-\epsilon}}{\epsilon}(e^{\epsilon r}-1)(\delta_{um}\delta_{nr}+\delta_{un}\delta_{mr}+\delta_{ur}\delta_{mn}-2\delta_{um}\delta_{mn}\delta_{nr}).$$

The correction matrix obtained for negative viscosity is

$$\begin{pmatrix}M'_1(\hat{k})\\ M'_2(\hat{k})\end{pmatrix}=\frac{1}{16}\frac{\alpha_c^2}{\beta_c^2}\frac{1}{\nu_c^2\eta_c^2}A_0A_d\frac{\Lambda^{-\epsilon}}{\epsilon}\times(e^{\epsilon r}-1)\begin{pmatrix}F'_1 & F'_2\\ F'_2 & F'_1\end{pmatrix}\begin{pmatrix}P^<_l(\hat{k})\\ Q^<_l(\hat{k})\end{pmatrix},$$

with

$$F'_1=4[(d^2-2d-16)+S(d+2)]\frac{\alpha_c^5}{\beta_c^3}+4[(d+y+12)+2S(-d^2+2d+14)]\frac{\alpha_c^4}{\beta_c^2}+4[(d^2-3d-16)-S(5d+2y+30)]\frac{\alpha_c^3}{\beta_c}+4[(3d+y+17)+2S(d+6)]\alpha_c^2-(4d+19+8S)\alpha_c\beta_c+4\beta_c^2-19\frac{\beta_c^3}{\alpha_c},$$

$$F'_2=4[-(d^2+2d+8)+S(d+3)]\frac{\alpha_c^5}{\beta_c^3}+4[(3d+y+16)+4S(d+4)]\frac{\alpha_c^4}{\beta_c^2}+4[3(d^2-d-8)-S(3d+2y+8)]\frac{\alpha_c^3}{\beta_c}+4[(d+y+13)+S(-2d^2+2d+7)]\alpha_c^2+2[-2(d+8)+15S]\alpha_c\beta_c+4(1-4S)\beta_c^2+2S\frac{\beta_c^3}{\alpha_c}.$$

Following the same procedure as described in Sec. V, we then obtain the differential equation for x (42).

The calculations for negative resistivity and positive viscosity are completely analogous to those of negative viscosity and positive resistivity. Owing to the symmetry of the poles, the results can be obtained from the above results. The poles are just reflected on the real axis, as compared with the case of negative viscosity and resistivity. Therefore the expression for the correction matrix is just changed by a minus sign, and so this change of sign is present in the differential equation for x, as shown in Eq. (43).

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